The geometry of the TDOA–based localization model

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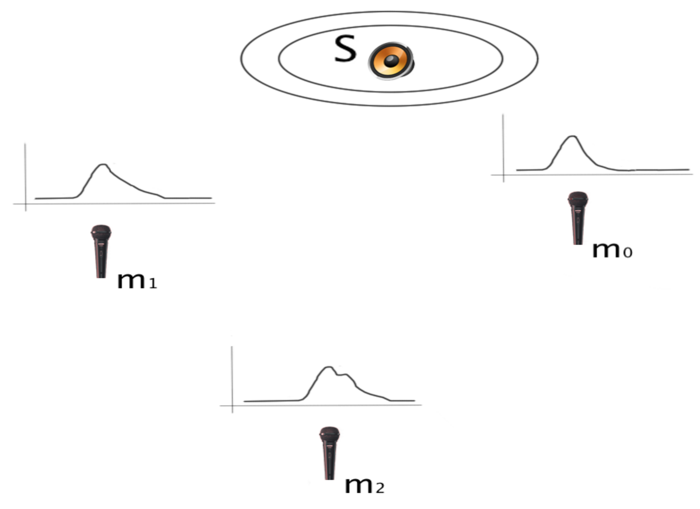
General Problem

Point-like (acoustic) source localization based on the time differences of arrival (TDOAs) of a signal to distinct receivers.

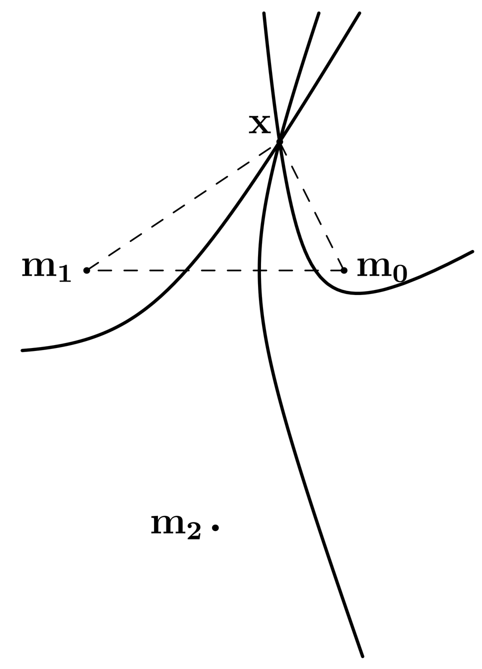


Experimental data: the TDOAs τ_{ji} of the signal to receivers $\mathrm{m_j}$ and $\mathrm{m_i}$, measured as the time shifts of the signal wavefront.

Goal: Obtain a complete description of the statistical model behind TDOA-based source localization, possibly with unsynchronized and uncalibrated receivers.

The Geometric Propagation Model

$$\mathrm{d_i(x)} = \mathrm{x} - \mathrm{m_i} \qquad d_i(\mathrm{x}) = \|\mathrm{d_i(x)}\|$$
$$\mathrm{d_{ji}} = \mathrm{m_j} - \mathrm{m_i} \qquad d_{ji} = \|\mathrm{d_{ji}}\|$$



$\hat{\tau}_{ji}$ = measured TDOA
ϵ_{ji} = measurement error
Propagation speed equal to 1.

$$\tau_{ji}(\mathrm{x}) = d_j(\mathrm{x}) - d_i(\mathrm{x})$$
$$\hat{\tau}_{ji} = \tau_{ji}(\mathrm{x}) + \epsilon_{ji}$$

$\tau_{ji}(\mathrm{x}) = \hat{\tau}_{ji}$ is an hyperbola branch with foci $\mathrm{m_i}, \mathrm{m_j}$ $\Rightarrow$ x is at the branches intersection.

- **Deterministic problem:** if $\epsilon_{ji} = 0$, conditions for existence and uniqueness of the localization.
- **Statistical problem:** if $\epsilon_{ji} \neq 0$, characterize the non linear model.

The GPS Problem

In the classical GPS problem one searches the location of a source in space using the **times of arrival t_i of signals (TOAs)** from n distinct satellites to the GPS receiver.

The TOA Model: $t_i(\mathrm{x}) = d_i(\mathrm{x}) + \epsilon_i + b$

- Because of the low accuracy of the receiver clock, one has to consider an additional bias b for each TOA.
- In order to eliminate b, one takes as input data the differences $t_i(\mathrm{x}) - t_1(\mathrm{x})$.

In the deterministic case, **the GPS problem reduces to the TDOA-based localization**.

- **Existence problem:** how many satellites are necessary to locate a source?
- **Uniqueness or Bifurcation problem:** in which cases is the localization unique?

Our Approach and Results

The TDOA Map

- a point–like source $\mathrm{x} \in \mathbb{R}^2$;
- $n+1$ synchronized and calibrated receivers $\mathrm{m_0}, \ldots, \mathrm{m_n} \in \mathbb{R}^2$;
- noiseless scenario, i.e. $\epsilon_{ji} = 0$.

There are only n independent data $\tau_{i0}(\mathrm{x})$.

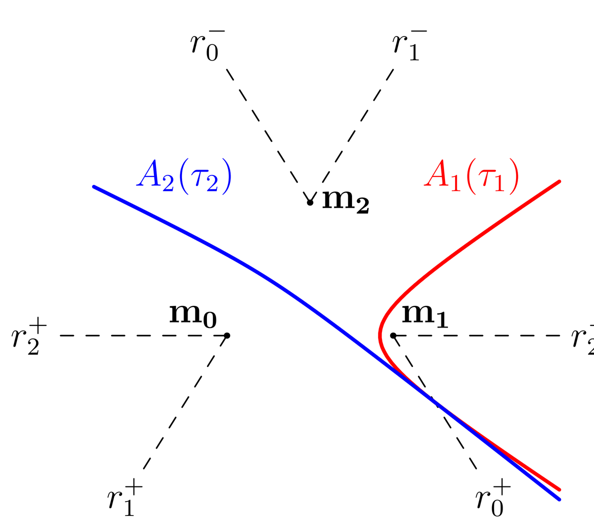
$$\tau_n : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$$
$$\mathrm{x} \longmapsto (\tau_{10}(\mathrm{x}), \ldots, \tau_{n0}(\mathrm{x}))$$

Let $\tau := (\tau_1, \ldots, \tau_n)$ be a measurement array:

- **Existence of localization** iff $\tau \in \mathrm{Im}(\tau_n)$;
- **Uniqueness of localization** iff $|\tau_n{}^{-1}(\tau)| = 1$.

$n = 2$ is the first case allowing the injectivity of τ_n.

The Local Analysis



$D := \cup_{i=0}^2 (r_i^- \cup r_i^+)$

The jacobian of τ_2 at $\mathrm{x} \neq \mathrm{m_0}, \mathrm{m_1}, \mathrm{m_2}$ has

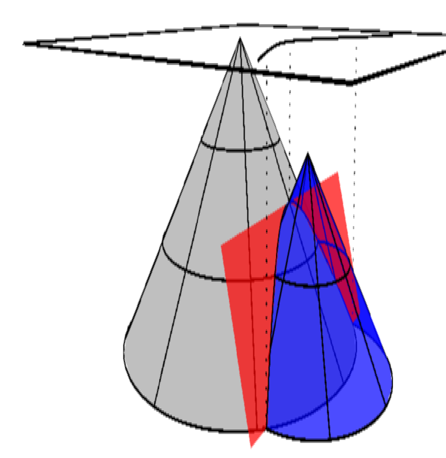
$$\mathrm{rk}(J) = \begin{cases} 1 & \text{if } \mathrm{x} \in D \\ 2 & \text{otherwise} \end{cases}$$

$A_i(\tau) := \{\mathrm{x} \in \mathbb{R}^2 | \tau_{i0}(\mathrm{x}) = \tau\}$, where $\tau \in \mathbb{R}$.

Proposition:
Assume $\mathrm{x} \in A_1(\tau_1) \cap A_2(\tau_2)$. Then, $A_1(\tau_1), A_2(\tau_2)$ meet transversally at x if, and only if, $\mathrm{x} \in \mathbb{R}^2 \setminus D$.

The Algebraic Global Analysis in $\mathbb{R}^{2,1}$

$$\mathrm{X} = (\mathrm{x}, \tau) \qquad \mathrm{D_i}(\mathrm{X}, \tau) = \mathrm{X} - \mathrm{M_i}(\tau)$$
$$\mathrm{M_i}(\tau) = (\mathrm{m_i}, \tau_i) \qquad \mathrm{D_{ji}}(\tau) = \mathrm{M_j}(\tau) - \mathrm{M_i}(\tau)$$



$$C_0^- : \begin{cases} \|\mathrm{D_0}\|^2 = 0 \\ \langle \mathrm{D_0}, \mathrm{e_3} \rangle \geq 0 \end{cases}$$

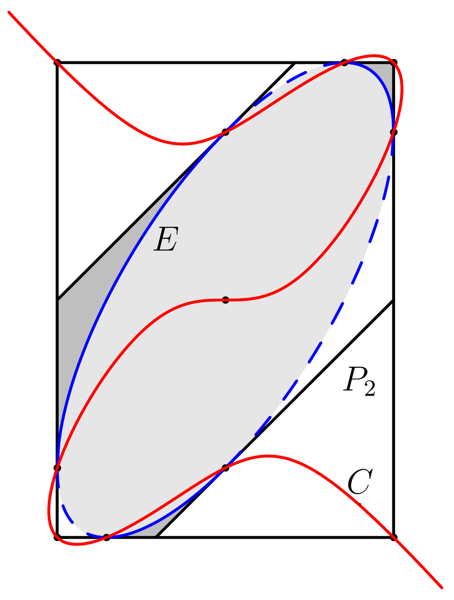
$$\Pi_i : \langle \mathrm{D_{i0}}, \mathrm{D_0} \rangle = \tfrac{1}{2} \|\mathrm{D_{i0}}\|^2$$

$$\pi(C_0^- \cap \Pi_i) = A_i$$

- Planes intersection: $L(\lambda) = \mathrm{L_0} + \lambda \mathrm{v}$, $\lambda \in \mathbb{R}$.
- Localization problem: $A_1 \cap A_2 = \pi(C_0^- \cap L) \Rightarrow$ $\|\mathrm{v}\|^2 \lambda^2 + 2\lambda \langle \mathrm{D_0(L_0)}, \mathrm{v} \rangle + \|\mathrm{D_0(L_0)}\|^2 = 0.$

The Image of τ_2

We are interested into the real negative solutions of the quadratic equation: **Descartes' rule of signs**.



$$a\lambda^2 + 2b\lambda + c = 0.$$

- $\Delta \geq 0$: the polytope P_2.
- $a = 0$: the ellipse E.
- $b = 0$: the cubic C through 11 points.
- $c = 0$: a quartic with only 4 real points.

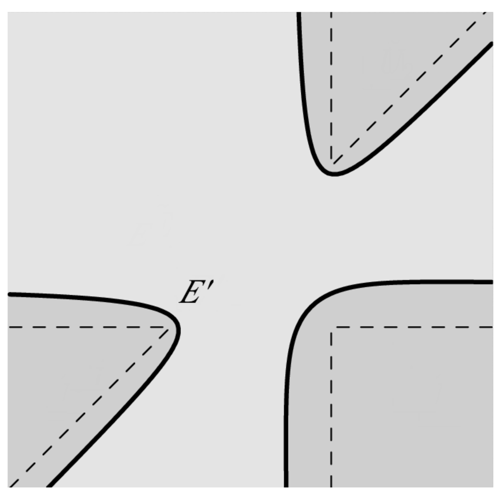
Theorem:

- τ_2 is $1:1$ on the light grey region;
- τ_2 is $2:1$ on the medium grey region.

The Bifurcation Problem

Given $\tau \in \mathrm{Im}(\tau_2)$ and a negative $\lambda(\tau)$:

$$\mathrm{x}(\tau) = \mathrm{L_0} + \lambda * ((\tau_2 \mathrm{d_{10}} - \tau_1 \mathrm{d_{20}}) \wedge \mathrm{e_3}).$$



Theorem:

- $\tau_2{}^{-1}(E)$ is the **bifurcation curve**;
- on E we have $\lambda(\tau) = -c(\tau)/2b(\tau)$, thus E' is a rational quintic, smooth on $\mathbb{R}^2$.

Theorem:

- the localization is unique on the light grey region;
- τ_2 defines a double cover on the medium grey region, D and ∂P_2 are the ramification and branching loci.

The Complete TDOA Map

In a noisy scenario we have to consider all the TDOAs.

$$\tau_2{}^* : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\mathrm{x} \longmapsto (\tau_{10}(\mathrm{x}), \tau_{20}(\mathrm{x}), \tau_{21}(\mathrm{x}))$$

$\mathrm{Im}(\tau_2{}^*)$ is contained into the plane

$$\mathcal{H} = \{\tau^* \in \mathbb{R}^3 \mid \tau_{10}^* + \tau_{20}^* - \tau_{21}^* = 0\}.$$

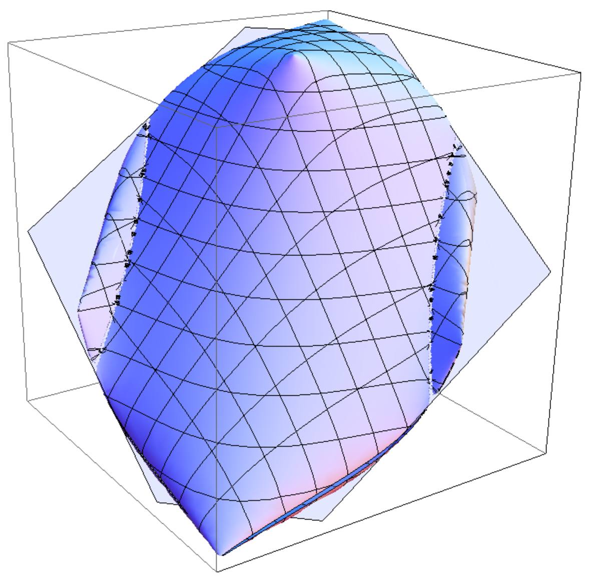
Let $p_i : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection forgetting the i-th coordinate. Then, we have:

$$\tau_2 = p_3 \circ \tau_2{}^* \quad \text{and}$$
$$p_3 : \mathrm{Im}(\tau_2{}^*) \longleftrightarrow \mathrm{Im}(\tau_2)$$

The description of $\mathrm{Im}(\tau_2{}^*)$ is the starting point for the study of the statistical model.

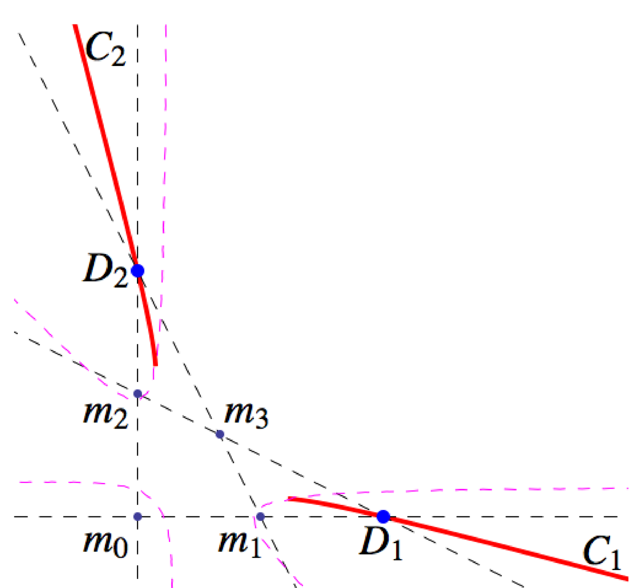
Work in Progress...

The Image of τ_3



- $\mathrm{Im}(\tau_3)$ is a subset of an algebraic sextic surface Σ.
- Σ is tangent to all the facets of the polytope P_3.
- Σ has many singular points and a singular locus S of dimension 1 on $\Sigma \cap \Pi$.

The Localization Problem



- On $C_1 \cup C_2 = \tau_3{}^{-1}(S)$ the TDOA map is $2:1$.
- The ramification locus is $D_1 \cup D_2$, where $\mathrm{rk}(J(\tau_3)) = 1$.

Some References

[1] B.Coll, J.J.Ferrando, J.A.Morales-Lladosaz, *Positioning systems in Minkowski space-time: from emission to inertial coordinates*, Class.Quant.Grav. 27, 065013 (2010).

[2] B.Coll, J.J.Ferrando, J.A.Morales-Lladosaz, *Positioning systems in Minkowski space-time: Bifurcation problem and observational data*, arXiv:1204.2241v2 [gr-qc].

[3] P.Bestagini, M.Compagnoni, F.Antonacci, A.Sarti, S.Tubaro, *TDOA-Based Acoustic Source Localization in the Space–Range Reference Frame*, to appear in Multidimensional Systems and Signal Processing.

[4] M.Compagnoni, R.Notari, F.Antonacci, A.Sarti, *The geometry of the TDOA–based source localization*, preprint.

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