The Geometry of the TDOA–based Source Localization

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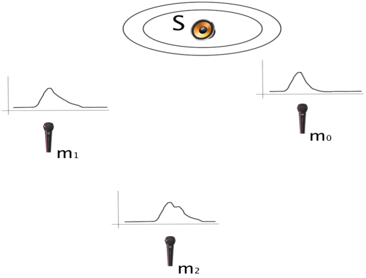
2D TDOA–based Localization

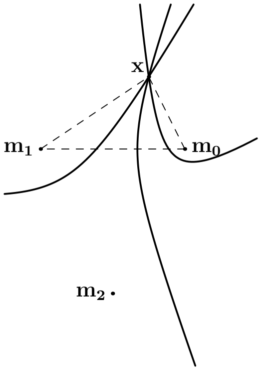
Problem: point-like (acoustic) source localization based on the time differences of arrival (TDOA) of a signal to distinct receivers lying on a plane.



Experimental data:
the TDOAs τ_{ji} of the signal to receivers $\mathbf{m_j}$ and $\mathbf{m_i}$, measured as the time shifts of the signal wavefront.

Goal: obtain a complete description of the statistical model behind TDOA-based source localization, possibly with unsynchronized and uncalibrated receivers.

The Geometric Propagation Model



$$\mathbf{d_i}(\mathbf{x}) = \mathbf{x} - \mathbf{m_i} \qquad d_i(\mathbf{x}) = \|\mathbf{d_i}(\mathbf{x})\|$$

$$\mathbf{d_{ji}} = \mathbf{m_j} - \mathbf{m_i} \qquad d_{ji} = \|\mathbf{d_{ji}}\|$$

$\hat{\tau}_{ji}$ = measured TDOA
ϵ_{ji} = measurement error
Propagation speed equal to 1.

$$\tau_{ji}(\mathbf{x}) = d_j(\mathbf{x}) - d_i(\mathbf{x})$$
$$\hat{\tau}_{ji} = \tau_{ij}(\mathbf{x}) + \epsilon_{ji}$$

$\tau_{ij}(\mathbf{x}) = \hat{\tau}_{ji}$ is an hyperbola branch with foci $\mathbf{m_i}, \mathbf{m_j}$ $\Rightarrow$ the source is at the branches intersection.

- **Deterministic problem:** if $\epsilon_{ji} = 0$, conditions for existence and uniqueness of the localization (the identifiability problem).
- **Statistical problem:** if $\epsilon_{ji} \neq 0$, characterize the non linear (and non algebraic) model.

The GPS Problem

In the classical GPS problem one searches the location of a source in space using the **times of arrival t_i of signals (TOAs)** from n distinct satellites to the GPS receiver.

The TOA Model: $t_i(\mathbf{x}) = d_i(\mathbf{x}) + \epsilon_i + b$

- Because of the low accuracy of the receiver clock, one has to consider an additional bias b for each TOA.
- In order to eliminate b, one chooses a reference satellite $\mathbf{m_1}$ and takes as input data the differences $t_i(\mathbf{x}) - t_1(\mathbf{x})$.

In the deterministic case **the GPS problem reduces to the TDOA-based localization**.

- **Existence problem:** how many satellites are necessary to locate a source?
- **Uniqueness or Bifurcation problem:** in which cases is the localization unique?

The TDOA Map

Hypothesis:

- a source $\mathbf{x} \in \mathbb{R}^2$;
- $n+1$ synchronized and calibrated receivers $\mathbf{m_0}, \ldots, \mathbf{m_n} \in \mathbb{R}^2$;
- noiseless scenario, i.e. $\epsilon_{ji} = 0$.

$$\tau_{ji}(\mathbf{x}) = \tau_{j0}(\mathbf{x}) - \tau_{i0}(\mathbf{x}) \;\Rightarrow\; n \text{ independent } \tau_{i0}(\mathbf{x}),\; i = 1, \ldots, n.$$

The TDOA map

$$\boldsymbol{\tau}_n : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^n \\ \mathbf{x} & \longmapsto & (\tau_{10}(\mathbf{x}), \ldots, \tau_{n0}(\mathbf{x})) \end{array}$$

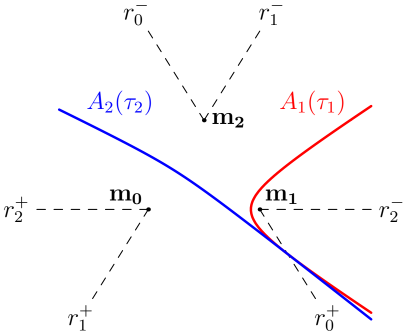
Given a measurements array $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$, we have:

- **Existence of localization** if, and only if, $\boldsymbol{\tau} \in \mathrm{Im}(\boldsymbol{\tau}_n)$, so the reduced **set of noiseless measurements is $\mathrm{Im}(\boldsymbol{\tau}_n)$.**
- **Uniqueness of localization** if, and only if, $|\boldsymbol{\tau}_n{}^{-1}(\boldsymbol{\tau})| = 1$.

The case $n = 2$ is the first one allowing the injectivity of $\boldsymbol{\tau}_n$.

The local analysis of $\boldsymbol{\tau_2}$

$\tau_{i0}(\mathbf{x}) \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{\mathbf{m_0}, \mathbf{m_1}, \mathbf{m_2}\})$ and $\nabla\tau_{i0}(\mathbf{x}) = \tilde{\mathbf{d}}_\mathbf{i}(\mathbf{x}) - \tilde{\mathbf{d}}_\mathbf{0}(\mathbf{x})$.



$D := \cup_{i=0}^2 (r_i^- \cup r_i^+)$

The jacobian of $\boldsymbol{\tau_2}$ at $\mathbf{x} \neq \mathbf{m_0}, \mathbf{m_1}, \mathbf{m_2}$ has

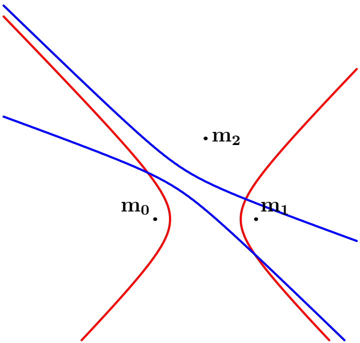
$$\mathrm{rk}(J) = \begin{cases} 1 & \text{if } \mathbf{x} \in D \\ 2 & \text{otherwise} \end{cases}$$

$A_i(\tau) := \{\mathbf{x} \in \mathbb{R}^2 | \, \tau_{i0}(\mathbf{x}) = \tau\}$, where $\tau \in \mathbb{R}$.

Proposition:
Assume $\mathbf{x} \in A_1(\tau_1) \cap A_2(\tau_2)$. Then, $A_1(\tau_1), A_2(\tau_2)$ meet transversally at $\mathbf{x}$ if, and only if, $\mathbf{x} \in \mathbb{R}^2 \setminus D$.

The Algebraic Global Analysis

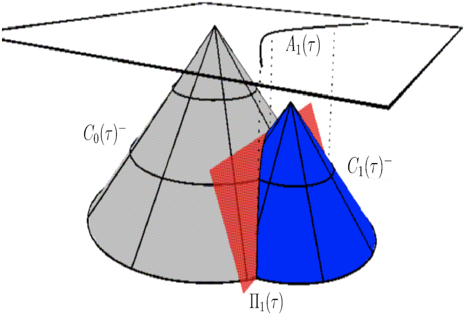
Let $\boldsymbol{\tau} = (\tau_1, \tau_2)$, we denote $A_i(\boldsymbol{\tau}) := A_i(\tau_i)$. We have
$\boldsymbol{\tau} \in$ **Im**$(\boldsymbol{\tau_2})$ **if, and only if,** $A_1(\boldsymbol{\tau}) \cap A_2(\boldsymbol{\tau}) \neq \emptyset$.



2D algebraic approach:

- Intersection of the two hyperbolas containing $A_1(\boldsymbol{\tau}), A_2(\boldsymbol{\tau})$.
- Problems: extra intersections, complex intersections.

The Algebraic Global Analysis



$$\begin{cases} \tau_1 = d_1(\mathbf{x}) - d_0(\mathbf{x}) \\ \tau_2 = d_2(\mathbf{x}) - d_0(\mathbf{x}) \end{cases} \Rightarrow$$

$$\begin{cases} \tau_1 - \tau = d_1(\mathbf{x}) \\ \tau_2 - \tau = d_2(\mathbf{x}) \\ \tau = -d_0(\mathbf{x}) \end{cases} \Rightarrow$$

$$\begin{cases} (\tau_1 - \tau)^2 = d_1(\mathbf{x})^2 \\ (\tau_2 - \tau)^2 = d_2(\mathbf{x})^2 \\ \tau^2 = d_0(\mathbf{x})^2 \end{cases}$$

3D algebraic approach:

- Intersection of three (half-)cones.
- Partially linear: the problem is equivalent to the intersection of a (half-)cone and two planes.
- No misleading solutions.

The 3D Minkowski Space

Notation:

$$\mathbf{X} = (\mathbf{x}, \tau) \qquad \mathbf{D_i}(\mathbf{X}, \boldsymbol{\tau}) = \mathbf{X} - \mathbf{M_i}(\boldsymbol{\tau})$$

$$\mathbf{M_i}(\boldsymbol{\tau}) = (\mathbf{m_i}, \tau_i) \qquad \mathbf{D_{ji}}(\boldsymbol{\tau}) = \mathbf{M_j}(\boldsymbol{\tau}) - \mathbf{M_i}(\boldsymbol{\tau})$$

The cones intersection:

$$\begin{cases} \|\mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau})\|^2 = 0 \\ \|\mathbf{D_i}(\mathbf{X}, \boldsymbol{\tau})\|^2 = 0 \end{cases} \Rightarrow \begin{cases} \|\mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau})\|^2 = 0 \\ \langle \mathbf{D_{i0}}(\boldsymbol{\tau}), \mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau}) \rangle = \frac{1}{2} \parallel \mathbf{D_{i0}}(\boldsymbol{\tau}) \parallel^2 \end{cases}$$

Let us define:

- $C_0(\boldsymbol{\tau}) = \{\mathbf{X} \in \mathbb{R}^{2,1} \mid \ \parallel \mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau}) \parallel^2 = 0\}$;
- $C_0(\boldsymbol{\tau})^- = \{\mathbf{X} \in C_0(\boldsymbol{\tau}) \mid \langle \mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau}), \mathbf{e_3} \rangle \geq 0\}$.
- $\Pi_i(\boldsymbol{\tau}) = \{\mathbf{X} \in \mathbb{R}^{2,1} \mid \langle \mathbf{D_{i0}}(\boldsymbol{\tau}), \mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau}) \rangle = \frac{1}{2} \parallel \mathbf{D_{i0}}(\boldsymbol{\tau}) \parallel^2\}$

Theorem

Let $\pi : \mathbb{R}^{2,1} \to \mathbb{R}^2$ be the projection onto the $\mathbf{x}$–plane. Then

$$\pi(C_0^- \cap \Pi_i(\boldsymbol{\tau})) = \begin{cases} A_i(\boldsymbol{\tau}) & \text{if } \tau_i \neq -d_{i0} \\ A_i(\boldsymbol{\tau}) \cup r_j^0 & \text{if } \tau_i = -d_{i0} \end{cases} \quad \text{with } i \neq j.$$

The Source Solution

Linear problem: $L(\boldsymbol{\tau}) = \Pi_1(\boldsymbol{\tau}) \cap \Pi_2(\boldsymbol{\tau})$ is a line for each $\boldsymbol{\tau} \in \mathbb{R}^2$, containing the point $\mathbf{L_0}(\boldsymbol{\tau})$ and parallel to $\mathbf{v}(\boldsymbol{\tau})$

$$\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})) = -\frac{*\left(\left(\|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2\mathbf{d_{20}} - \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2\mathbf{d_{10}}\right) \wedge \mathbf{e_3}\right)}{2\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|}$$

$$\mathbf{v}(\boldsymbol{\tau}) = *(\mathbf{D_{10}}(\boldsymbol{\tau}) \wedge \mathbf{D_{20}}(\boldsymbol{\tau})) = *((\mathbf{d_{10}} \wedge \mathbf{d_{20}}) + (\tau_2\mathbf{d_{10}} - \tau_1\mathbf{d_{20}}) \wedge \mathbf{e_3}).$$

Quadratic problem: $A_1(\boldsymbol{\tau}) \cap A_2(\boldsymbol{\tau}) \subseteq \pi(C_0^- \cap L(\boldsymbol{\tau}))$. Hence, we study $\|\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})) + \lambda\mathbf{v}(\boldsymbol{\tau})\|^2 = 0$, or, explicitly,

$$\|\mathbf{v}(\boldsymbol{\tau})\|^2\lambda^2 + 2\lambda\langle \mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})), \mathbf{v}(\boldsymbol{\tau})\rangle + \|\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau}))\|^2 = 0.$$

This equation in $\lambda \in \mathbb{R}$ has degree at most 2, with coefficients depending on $\boldsymbol{\tau}$.

The Source Solution Analysis

By setting:

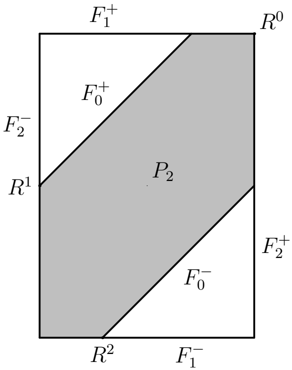
- $a(\boldsymbol{\tau}) = \|\mathbf{v}(\boldsymbol{\tau})\|^2 = \|\tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}\|^2 - \|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2$
- $b(\boldsymbol{\tau}) = \langle \mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})), \mathbf{v}(\boldsymbol{\tau})\rangle = \\ = \dfrac{\langle \tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}, \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} - \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}}\rangle}{2\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|}$
- $c(\boldsymbol{\tau}) = \|\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau}))\|^2 = \dfrac{\left\| \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}} - \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} \right\|^2}{4\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2}$

$$\Rightarrow \qquad a(\boldsymbol{\tau})\lambda^2 + 2b(\boldsymbol{\tau})\lambda + c(\boldsymbol{\tau}) = 0.$$

We are interested into the real negative solutions, therefore we use **Descartes' rule of signs** to characterize $\mathrm{Im}(\boldsymbol{\tau_2})$.

The **coefficients are polynomials** with respect to τ_1, τ_2.

The Polytope



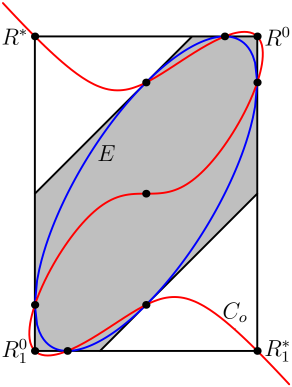
$$\begin{cases} -d_{10} \le \tau_1 \le d_{10} \\ -d_{20} \le \tau_2 \le d_{20} \\ -d_{21} \le \tau_2 - \tau_1 \le d_{21} \end{cases}$$

The six inequalities define a **polygon** P_2, i.e. a two dimensional convex polytope. P_2 has **six facets** $F_k^{\pm}$.

$$P_2 = \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid \|D_{ji}(\boldsymbol{\tau})\|^2 \ge 0,\ \forall\, i,j\}$$

- $\text{Im}(\boldsymbol{\tau_2}) \subsetneq P_2$, in particular $\boldsymbol{\tau_2}^{-1}(F_k^{\pm}) = r_k^{\pm}$ and $\boldsymbol{\tau_2}^{-1}(R^k) = \mathbf{m_k}$.
- $\Delta(\boldsymbol{\tau}) = b(\boldsymbol{\tau})^2 - 4a(\boldsymbol{\tau})c(\boldsymbol{\tau}) = 0$ is a sextic algebraic curve in the $\boldsymbol{\tau}$–plane, and it factors as the six lines supporting $F_k^{\pm}$.
- $\Delta > 0$ on $\mathring{P}_2$.

The analysis of the coefficients



$$c = \frac{\left\| \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}} - \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} \right\|^2}{4\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2}$$

- $c(\boldsymbol{\tau}) = 0$ iff $\boldsymbol{\tau} \in \{R^0, R^*, R_1^*, R_1^0\}$,
- $c(\boldsymbol{\tau}) > 0$ otherwise.

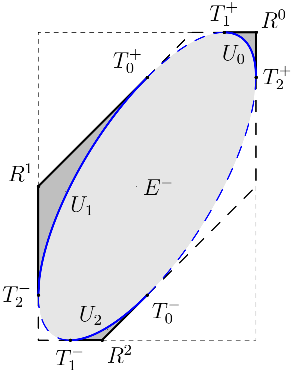
$$a = \|\tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}\|^2 - \|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2$$

- $a = 0$ is the unique ellipse E tangent to each facet of P_2,
- $a < 0$ inside E and $a > 0$ outside.

$$b(\boldsymbol{\tau}) = \frac{\langle \tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}, \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} - \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}} \rangle}{2\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|}$$

- $b = 0$ is the unique cubic C through the 11 marked points,
- only the odd circuit C_o of C contains the 11 points, while the even circuit C_e (if it exists) does not intersect P_2.

The Image of τ_2



- On the light gray region E^- we have $a < 0$ and $c > 0$.
- On the medium gray region $U = U_0 \cup U_1 \cup U_2$ we have $a, b, c > 0$.

Theorem

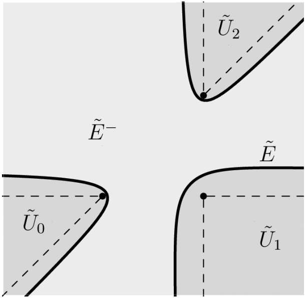
- $Im(\tau_2) = E^- \cup \bar{U} \setminus \{T_0^\pm, T_1^\pm, T_2^\pm\}$
- $|\tau_2^{-1}(\tau)| = \begin{cases} 2 & \text{if } \tau \in U \\ 1 & \text{if } \tau \in Im(\tau_2) \setminus U \end{cases}$

- $\tau \in \partial P_2 \cap \mathrm{Im}(\tau_2)$: $L(\tau), C_0^-$ and $A_1(\tau), A_2(\tau)$ meet tangentially.
- $\tau \in E$: $L(\tau)$ is parallel to a generatrix of C_0 and $A_1(\tau), A_2(\tau)$ have one parallel asymptote.
- $\tau \in E^-$: $L(\tau)$ intersects both C_0^-, C_0^+ and $|A_1(\tau) \cap A_2(\tau)| = 1$.
- $\tau \in U$: $L(\tau)$ intersects twice C_0^- and $|A_1(\tau) \cap A_2(\tau)| = 2$.

The Bifurcation Problem

Given $\boldsymbol{\tau} \in \mathrm{Im}(\boldsymbol{\tau_2})$ and a negative solution $\lambda(\boldsymbol{\tau})$:

$$\mathbf{x}(\boldsymbol{\tau}) = \mathbf{L_0} + \lambda * ((\tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}) \wedge \mathbf{e_3}).$$



Theorem:

- $\tilde{E} = \boldsymbol{\tau_2}^{-1}(E)$ is the **bifurcation curve**, separating the 1:1 and 2:1 regions of $\boldsymbol{\tau_2}$;
- on E we have $\lambda(\boldsymbol{\tau}) = -c(\boldsymbol{\tau})/2b(\boldsymbol{\tau})$, thus $\tilde{E}$ is a rational quintic, smooth on $\mathbb{R}^2$.

- The localization is unique on light grey region $\tilde{E}^- = \boldsymbol{\tau_2}^{-1}(E^-)$;
- $\boldsymbol{\tau_2}$ is a double cover on medium grey region $\tilde{U}_0 \cup \tilde{U}_1 \cup \tilde{U}_2 = \boldsymbol{\tau_2}^{-1}(U)$, where D and ∂P_2 are the ramification and branching loci.
- As $\boldsymbol{\tau}$ approaches to ∂P_2, $\boldsymbol{\tau_2}^{-1}(\boldsymbol{\tau})$ converges to a point on D. As $\boldsymbol{\tau}$ approaches to E, $\boldsymbol{\tau_2}^{-1}(\boldsymbol{\tau})$ converges to a point on $\tilde{E}$ and to another at infinity.

The Complete TDOA Map

In a noisy scenario we have to consider all the TDOAs.

$$\begin{array}{rccc} \boldsymbol{\tau_2}^* : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & \mathbf{x} & \longmapsto & (\tau_{10}(\mathbf{x}), \tau_{20}(\mathbf{x}), \tau_{21}(\mathbf{x})) \end{array}$$

The **set of noiseless measurements is Im$(\boldsymbol{\tau_2}^*)$.** It is contained into the plane

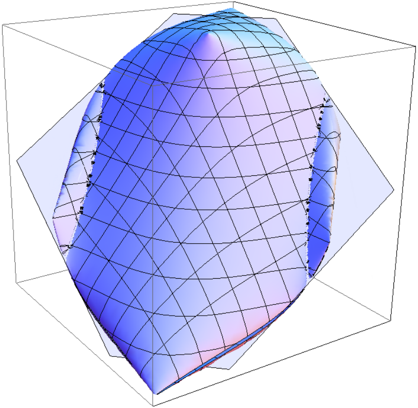
$$\mathcal{H} = \{\boldsymbol{\tau}^* \in \mathbb{R}^3 \mid \tau_{10}^* + \tau_{20}^* - \tau_{21}^* = 0\}.$$

Let $p_i : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection forgetting the i-th coordinate. Then, we have:

$$\boldsymbol{\tau_2} = p_3 \circ \boldsymbol{\tau_2}^* \quad \text{and} \quad p_3 : \mathrm{Im}(\boldsymbol{\tau_2}^*) \longleftrightarrow \mathrm{Im}(\boldsymbol{\tau_2})$$

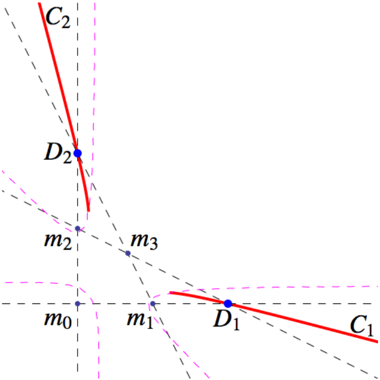
The description of the measurements set Im$(\boldsymbol{\tau_2}^*)$ is the starting point for the study of the statistical model.

The Image of τ_3



- $\mathrm{Im}(\tau_3)$ is a semi–algebraic set contained in a sextic surface Σ.
- Σ is tangent to all the facets of the polytope P_3.
- Σ has many singular points and a singular locus on a conic S contained in the plane Π.

The Localization Problem



- The TDOA map τ_3 is a homeomorphism if, and only if, the convex hull of $\mathbf{m_0}, \dots, \mathbf{m_3}$ is a triangle.
- If the convex hull is a quadrangle, there are two 1D sets C_1, C_2 where the TDOA map is $2:1$. We have $C_1 \cup C_2 = {\tau_3}^{-1}(S)$.
- The ramification locus is $D_1 \cup D_2$, where $\mathrm{rk}(J(\tau_3)) = 1$.

Conclusions and Perspectives

In this work:

- we studied the planar TDOA-based localization problem with three receivers in a noiseless scenario;
- in particular we have characterized the measurements space and the bifurcation curve in terms of real (semi)algebraic sets;
- we introduced the complete measurements space.

In future works we will:

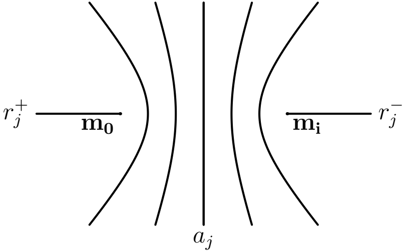
- complete the cases $n \geq 3$;
- study the 3-dimensional TDOA-based localization;
- study the statistical properties of the model.

Bibliography

- M.Compagnoni, P.Bestagini, F.Antonacci, A.Sarti, S.Tubaro, *Localization of Acoustic Sources Through the Fitting of Propagation Cones Using Multiple Independent Arrays*, IEEE Transactions on Audio, Speech, and Language Processing, Vol. 20 (2012), Issue 7, 1964–1975.
- P.Bestagini, M.Compagnoni, F.Antonacci, A.Sarti, S.Tubaro, *TDOA-Based Acoustic Source Localization in the Space–Range Reference Frame*, to appear in Multidimensional Systems and Signal Processing.
- B.Coll, J.J.Ferrando, J.A.Morales-Lladosaz, *Positioning systems in Minkowski space-time: from emission to inertial coordinates*, Class.Quant.Grav. 27, 065013 (2010).
- B.Coll, J.J.Ferrando, J.A.Morales-Lladosaz, *Positioning systems in Minkowski space-time: Bifurcation problem and observational data*, arXiv:1204.2241v2 [gr-qc].

Geometric Interpretation

$A_i(\tau) := \{\mathbf{x} \in \mathbb{R}^2 | \, \tau_i(\mathbf{x}) = \tau, \tau \in \mathbb{R}\}$ is the level set of $\tau_i(\mathbf{x})$.



- If $|\tau| > d_{i0}$, then $A_i(\tau) = \emptyset$.
- If $0 < |\tau| < d_{i0}$, then $A_i(\tau)$ is the branch of hyperbola with foci $\mathbf{m_0}, \mathbf{m_i}$ and parameter τ.
- $A_i(\tau) = \begin{cases} r_j^+ & \text{if } \tau = d_{i0} \\ r_j^- & \text{if } \tau = -d_{i0} \\ a_j & \text{if } \tau = 0 \end{cases}$

The 3D Minkowski Space

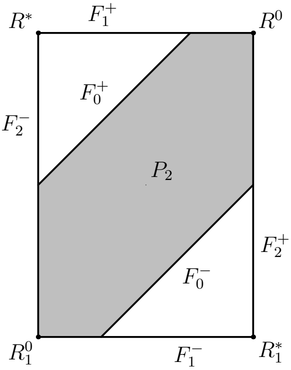
We take:

- V a 3–dimensional $\mathbb{R}$–vector space and $\wedge V$ its exterior algebra;
- $b : V \times V \to \mathbb{R}$ a non–degenerate, symmetric bilinear form with signature $(+\ +\ -)$;
- $B = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ an orthonormal basis.

Then:

- $\langle \mathbf{u}, \mathbf{v} \rangle = b(\mathbf{u}, \mathbf{v}) = \langle \sum_{i=1}^{3} u_i \mathbf{e_i}, \sum_{j=1}^{3} v_i \mathbf{e_i} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3$;
- $\| \mathbf{u} \|^2 = b(\mathbf{u}, \mathbf{u}) = \| \sum_{i=1}^{3} u_i \mathbf{e_i} \|^2 = u_1^2 + u_2^2 - u_3^2$;
- $\langle \mathbf{u_1} \wedge \cdots \wedge \mathbf{u_k}, \mathbf{v_1} \wedge \cdots \wedge \mathbf{v_k} \rangle = \det \begin{pmatrix} \langle \mathbf{u_1}, \mathbf{v_1} \rangle & \dots & \langle \mathbf{u_1}, \mathbf{v_k} \rangle \\ \vdots & & \vdots \\ \langle \mathbf{u_k}, \mathbf{v_1} \rangle & \dots & \langle \mathbf{u_k}, \mathbf{v_k} \rangle \end{pmatrix}$;
- $(\mathbf{e_1} \wedge \mathbf{e_2}, \mathbf{e_1} \wedge \mathbf{e_3}, \mathbf{e_2} \wedge \mathbf{e_3})$ is an orthonormal basis of $\wedge^2 V$ with signature $(+\ -\ -)$;
- $\boldsymbol{\omega} := \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$ is an orthonormal basis of $\wedge^3 V$ with $\| \boldsymbol{\omega} \|^2 = -1$;
- $* : \wedge^k V \to \wedge^{3-k} V$ defined as $\mathbf{x} \wedge *\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle \boldsymbol{\omega}$.

The Quartic

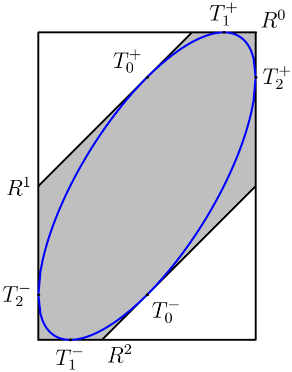


$$c(\boldsymbol{\tau}) = \|\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau}))\|^2 = \frac{\left\| \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}} - \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} \right\|^2}{4\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2}$$

Proposition

$c(\boldsymbol{\tau})$ is a degree four polynomial in (τ_1, τ_2) and:

- *$c(\boldsymbol{\tau}) = 0$ iff $\boldsymbol{\tau} \in \{R^0, R^*, R_1^*, R_1^0\}$, otherwise $c(\boldsymbol{\tau}) > 0$.*
- *$\nabla c(\boldsymbol{\tau})$ vanishes at R^0, R^*, R_1^*, R_1^0.*
- *In $\mathbb{P}^2_{\mathbb{C}}$, $c(\boldsymbol{\tau}) = 0$ is a quartic algebraic curve with four (real) singular points, and so it factors as two conics.*



The Ellipse

$$a(\boldsymbol{\tau}) = \|\mathbf{v}(\boldsymbol{\tau})\|^2 = \|\tau_2\mathbf{d_{10}} - \tau_1\mathbf{d_{20}}\|^2 - \|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|^2$$

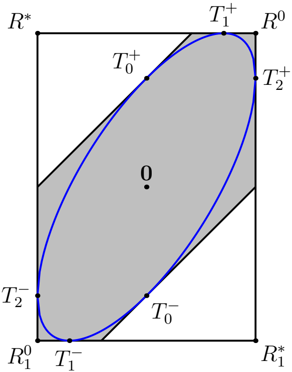
- $E := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid a(\boldsymbol{\tau}) = 0\}$;
- $E^+ := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid a(\boldsymbol{\tau}) > 0\}$;
- $E^- := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid a(\boldsymbol{\tau}) < 0\}$.

Proposition

$a(\boldsymbol{\tau})$ is a degree two polynomial in (τ_1, τ_2) and:

- *$E \subset P_2$ is a smooth ellipse with center at $\mathbf{0}$.*
- *E is the unique conic tangent to each facet of P_2.*
- *E^- is the connected component of $\mathbb{R}^2 \setminus E$ containing $\mathbf{0}$.*

The Cubic



$$b(\boldsymbol{\tau}) = \langle \mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})), \mathbf{v}(\boldsymbol{\tau}) \rangle = \frac{\langle \tau_2 \mathbf{d_{10}} - \tau_1 \mathbf{d_{20}}, \|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{d_{10}} - \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{d_{20}} \rangle}{2\|\mathbf{d_{10}} \wedge \mathbf{d_{20}}\|}$$

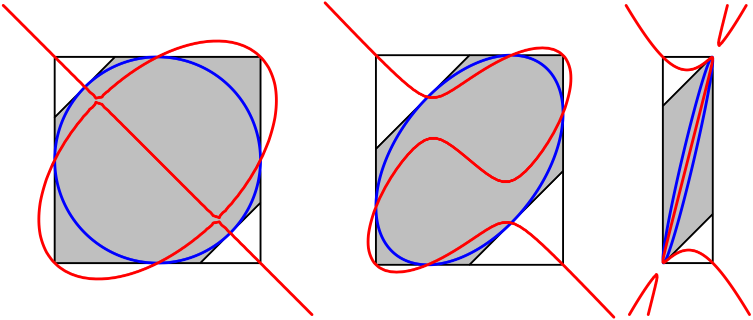
- $C := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid b(\boldsymbol{\tau}) = 0\}$;
- $C^+ := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid b(\boldsymbol{\tau}) > 0\}$;
- $C^- := \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid b(\boldsymbol{\tau}) < 0\}$.

Proposition

$b(\boldsymbol{\tau})$ is a degree three polynomial in (τ_1, τ_2) and:

- *C is the unique cubic curve containing the points $T_0^\pm, T_1^\pm, T_2^\pm, R^0, R_1^0, R^*, R_1^*, \mathbf{0}$.*
- *C is a smooth curve, unless $d_{10} = d_{20}$. In this case, C is the union of a line and a conic.*

The Cubic



- C is a cubic curve with 2–fold rotational symmetry w.r.t. $\mathbf{0}$, which is an inflectional point if C is smooth.
- C intersects transversally E and the lines supporting ∂P_2.
- The tangent to C at R^0, R_1^0, R^*, R_1^* are orthogonal to $F_0^{\pm}$.

Proposition

If C is smooth, the points $T_0^{\pm}, T_1^{\pm}, T_2^{\pm}, R^0, R^*, R_1^0, R_1^*, \mathbf{0}$ belong to the odd circuit C_o of C, while the even circuit C_e (if it exists) does not intersect P_2.

The Quintic

Given $\boldsymbol{\tau} \in \mathrm{Im}(\boldsymbol{\tau}_2)$ and a negative solution $\lambda(\boldsymbol{\tau})$ of the quadratic equation, on the **x**–plane we have

$$\mathbf{x}(\boldsymbol{\tau}) = \mathbf{L}_0(\boldsymbol{\tau}) + \lambda(\boldsymbol{\tau}) * ((\tau_2 \mathbf{d}_{10} - \tau_1 \mathbf{d}_{20}) \wedge \mathbf{e}_3).$$

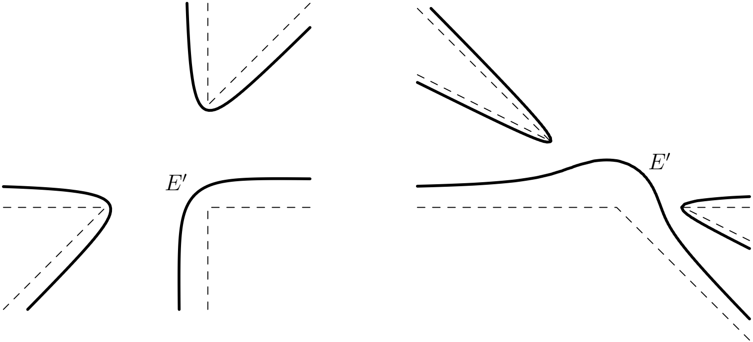
The preimage $E' = {\boldsymbol{\tau}_2}^{-1}(E)$ of the ellipse is the **bifurcation curve**, which separates the single and double preimage regions.

- On E we have $a(\boldsymbol{\tau}) = 0$, thus $\lambda(\boldsymbol{\tau}) = -c(\boldsymbol{\tau})/2b(\boldsymbol{\tau})$.
- Because of the symmetry, $\mathbf{x}(\boldsymbol{\tau})$ defines a $2:1$ map $E \to E'$.
- By "parametrizing" E via the pencil of lines through $\mathbf{0}$, we obtain a parametric representation of E' given as ratios of degree 5 polynomials without common factors.

Theorem

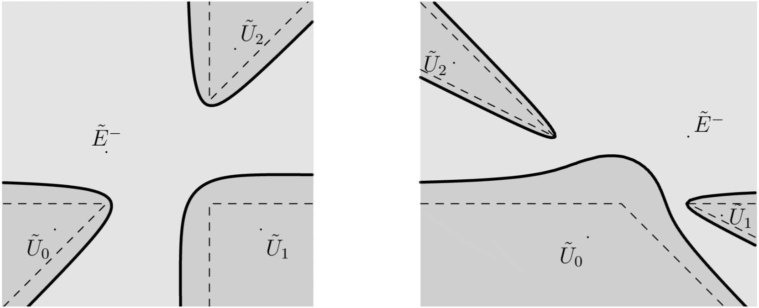
E' is a rational degree 5 curve, whose ideal points are the ones of the lines r_0, r_1, r_2, and the two ones of E.

The Quintic



- E' on $\mathbb{R}^2$ consists of **three disjoint unbounded arcs**, one for each arc of $E \cap \mathrm{Im}(\tau_2)$, with $\mathbf{m_0}, \mathbf{m_1}, \mathbf{m_2} \notin E'$.
- E' has **no self–intersections** and it is **regularly parameterized**.
- In $\mathbb{P}^2_{\mathbb{C}}$, the rational quintic curve has singular points.

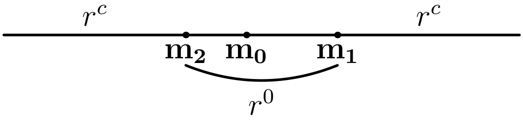
The Bifurcation Problem



- $\tilde{E}^- = \boldsymbol{\tau_2}^{-1}(E^-)$, $\tilde{U}_i = \boldsymbol{\tau_2}^{-1}(U_i)$ are open subsets separate by E'.
- $\boldsymbol{\tau_2}$ is 1–to–1 on $\tilde{E}^-$.
- $\tilde{U}_i$ has two connected components separeted by $r_j^{(\pm)}, r_k^{(\pm)}$, and $\boldsymbol{\tau_2}$ is 1–to–1 on each of them.
- As $\boldsymbol{\tau}$ approaches to ∂P_2, $\boldsymbol{\tau_2}^{-1}(\boldsymbol{\tau})$ converges to a point on $r_j^{\pm} \cup r_k^{\pm}$.
- As $\boldsymbol{\tau}$ approaches to E, $\boldsymbol{\tau_2}^{-1}(\boldsymbol{\tau})$ converges to a point on E' and to another at ∞.

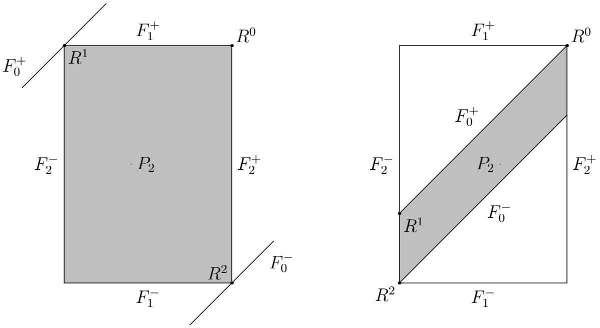
Special Configurations I

Assume that $\mathbf{m_0}, \mathbf{m_1}, \mathbf{m_2}$ are contained in the straight line r. Let r^0 the smallest line segment containing all the three points, and r^c its complement in r.



- $\text{rk}(J(\mathbf{x})) = \begin{cases} 0 & \text{if } \mathbf{x} \in r^c \\ 1 & \text{if } \mathbf{x} \in r^0 \\ 2 & \text{otherwise} \end{cases}$.
- If $\mathbf{x} \in A_1(\boldsymbol{\tau}) \cap A_2(\boldsymbol{\tau})$, then $A_1(\boldsymbol{\tau}) \cap A_2(\boldsymbol{\tau})$ is finite if, and only if, $\mathbf{x} \in \mathbb{R}^2 \setminus r^c$.
- $A_1(\boldsymbol{\tau})$ and $A_2(\boldsymbol{\tau})$ meet transversally at $\mathbf{x}$ if, and only if, $\mathbf{x} \in \mathbb{R}^2 \setminus r$.

Special Configurations II



$$\begin{cases} -d_{10} \leq \tau_1 \leq d_{10} \\ -d_{20} \leq \tau_2 \leq d_{20} \\ -d_{21} \leq \tau_2 - \tau_1 \leq d_{21} \end{cases}$$

There are two redundant inequalities, therefore the polygon P_2 has only **four facets**.

In the following we assume that $\mathbf{m_0}$ is between $\mathbf{m_1}$ and $\mathbf{m_2}$, that corresponds to the first polytope.

Special Configurations III

Linear problem: $L(\boldsymbol{\tau}) = \Pi_1(\boldsymbol{\tau}) \cap \Pi_2(\boldsymbol{\tau})$. Then:

- $L(\boldsymbol{\tau}) = \emptyset$ if, and only if, $d_{10}\tau_2 + d_{20}\tau_1 = 0$.
- $L(\boldsymbol{\tau}) = \Pi_1(\boldsymbol{\tau}) = \Pi_2(\boldsymbol{\tau})$ if, and only if, $\boldsymbol{\tau} = (\pm d_{10}, \mp d_{20})$.
- $L(\boldsymbol{\tau})$ is a line parallel to the **x**–plane otherwise, with

$$\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})) = \frac{*\left(\mathbf{v}(\boldsymbol{\tau}) \wedge \left(\|\mathbf{D_{20}}(\boldsymbol{\tau})\|^2 \mathbf{D_{10}}(\boldsymbol{\tau}) - \|\mathbf{D_{10}}(\boldsymbol{\tau})\|^2 \mathbf{D_{20}}(\boldsymbol{\tau})\right)\right)}{2d_{10}^2(d_{10}\tau_2 + d_{20}\tau_1)}$$

$$\mathbf{v}(\boldsymbol{\tau}) = *(\mathbf{d_{10}} \wedge \mathbf{e_3})$$

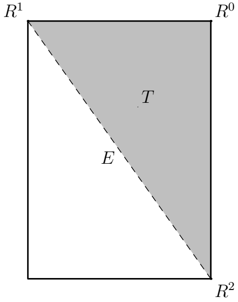
Quadratic problem: $\|\mathbf{v}(\boldsymbol{\tau})\|^2 > 0$, $\langle \mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})), \mathbf{v}(\boldsymbol{\tau})\rangle = 0$, then

$$\|\mathbf{v}(\boldsymbol{\tau})\|^2 \lambda^2 + \|\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau}))\|^2 = 0.$$

The line $L(\boldsymbol{\tau})$ intersect only one half–cones C_0^+, C_0^-:

$$\langle \mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})), \mathbf{e_3}\rangle > 0.$$

Special Configurations IV



Let E be the open segment with endpoints R^1, R^2 and T the triangle with side E and vertex R^0.

Theorem

$Im(\boldsymbol{\tau_2}) = T \setminus E$

$$|\boldsymbol{\tau_2}^{-1}(\boldsymbol{\tau})| = \begin{cases} \infty & \textit{if } \boldsymbol{\tau} \in \partial E \\ 2 & \textit{if } \boldsymbol{\tau} \in \mathring{T} \\ 1 & \textit{otherwise} \end{cases}$$

- $\boldsymbol{\tau} \in E$: $\Pi_1(\boldsymbol{\tau}), \Pi_2(\boldsymbol{\tau})$ are parallel and $A_1(\boldsymbol{\tau}), A_2(\boldsymbol{\tau})$ have parallel asymptotes.
- $\boldsymbol{\tau} \in \partial E$: $L(\boldsymbol{\tau}) = \Pi_1(\boldsymbol{\tau}) = \Pi_2(\boldsymbol{\tau})$ and $A_1(\boldsymbol{\tau}) \cap A_2(\boldsymbol{\tau}) = r^c$.
- $\boldsymbol{\tau} \in \partial T \setminus \bar{E}$: $L(\boldsymbol{\tau})$ is tangent to C_0^- and $A_1(\boldsymbol{\tau}), A_2(\boldsymbol{\tau})$ intersect at one point on r^0, with double multiplicity.
- $\boldsymbol{\tau} \in \mathring{T}$: $L(\boldsymbol{\tau})$ intersects C_0^- and $A_1(\boldsymbol{\tau}), A_2(\boldsymbol{\tau})$ intersect at two points symmetric w.r.t. the line r.

Restoring the Symmetry I

In the definition of the TDOA map we chose $\mathbf{m_0}$ as reference receiver, breaking the symmetry of the problem.

$$\mathbf{D_j}(\mathbf{X}, \boldsymbol{\tau}) = \mathbf{D_0}(\mathbf{X}, \boldsymbol{\tau}) + \mathbf{D_{0j}}(\boldsymbol{\tau}) \qquad \mathbf{D_{ij}}(\boldsymbol{\tau}) = \mathbf{D_{i0}}(\boldsymbol{\tau}) + \mathbf{D_{0j}}(\boldsymbol{\tau})$$

Theorem

$$\pi(C_0^-(\boldsymbol{\tau}) \cap C_1^-(\boldsymbol{\tau}) \cap C_2^-(\boldsymbol{\tau})) = \pi(C_i^-(\boldsymbol{\tau}) \cap \Pi_{ji}(\boldsymbol{\tau}) \cap \Pi_{ki}(\boldsymbol{\tau}))$$

In particular the three lines $L_0(\boldsymbol{\tau}), L_1(\boldsymbol{\tau}), L_2(\boldsymbol{\tau})$ coincide.

$$\mathbf{v_0}(\boldsymbol{\tau}) = \mathbf{v_1}(\boldsymbol{\tau}) = \mathbf{v_2}(\boldsymbol{\tau}).$$

$$\mathbf{D_0}(\mathbf{L_0}(\boldsymbol{\tau})) \neq \mathbf{D_1}(\mathbf{L_1}(\boldsymbol{\tau})) \neq \mathbf{D_2}(\mathbf{L_2}(\boldsymbol{\tau})).$$

The localization does not depend on the choice of the reference receiver. What does it happen to the $\mathrm{Im}(\boldsymbol{\tau_2})$ in the $\boldsymbol{\tau}$–space?

Restoring the Symmetry II

The complete TDOA map

$$\boldsymbol{\tau_2}^* : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ \mathbf{x} & \longmapsto & (\tau_{10}(\mathbf{x}), \tau_{20}(\mathbf{x}), \tau_{21}(\mathbf{x})) \end{array}$$

- $\mathcal{H} = \{\boldsymbol{\tau}^* \in \mathbb{R}^3 \mid \mathbf{D_{01}}(\boldsymbol{\tau}^*) + \mathbf{D_{12}}(\boldsymbol{\tau}^*) + \mathbf{D_{20}}(\boldsymbol{\tau}^*) = \mathbf{0}\}$;
- $\mathcal{P}_2 = \{\boldsymbol{\tau}^* \in \mathcal{H} \mid \|\mathbf{D_{ji}}(\boldsymbol{\tau}^*)\|^2 \geq 0 \text{ for every } i, j\}$;
- $\mathcal{E} = \{\boldsymbol{\tau}^* \in \mathcal{H} \mid \|\mathbf{v_0}(\boldsymbol{\tau}^*)\|^2 = 0\}$;
- $\mathcal{C}_i = \{\boldsymbol{\tau}^* \in \mathcal{H} \mid \langle \mathbf{D_i}(\mathbf{L_i}(\boldsymbol{\tau}^*), \mathbf{v_i}(\boldsymbol{\tau}^*)\rangle = 0\}$.

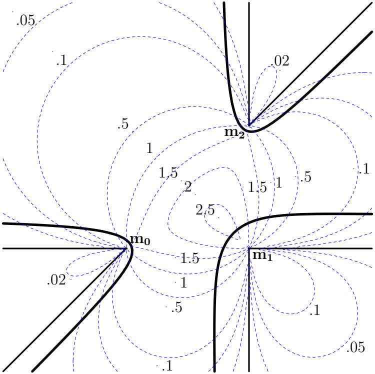
Theorem

Let $p_i : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the projection forgetting the i-th coordinate.

- *$\mathcal{H}$ is a plane containing the admissible TDOA triples;*
- *$\mathcal{P}_2$ is a polygon such that $p_3(\mathcal{P}_2) = P_2$;*
- *$\mathcal{E}$ is the ellipse tangent to all the sides of $\mathcal{P}_2$ and $p_3(\mathcal{E}) = E$;*
- *$\mathcal{C}_i$ is the cubic curve containing $\mathcal{E} \cap \partial\mathcal{P}_2, \mathcal{R}^i, \mathcal{R}^i_0, \mathcal{R}^{i*}, \mathcal{R}^{i*}_1, \mathbf{0}$.*

$$\boldsymbol{\tau_2} = p_3 \circ \boldsymbol{\tau_2}^* \quad \textit{and} \quad p_3 : Im(\boldsymbol{\tau_2}^*) \longleftrightarrow Im(\boldsymbol{\tau_2})$$

The accuracy of the localization



- $|\det(J(\mathbf{x}))|$ is the ratio between the areas of two corresponding infinitesimal regions in the τ and in the x planes. At first order, the accuracy is best in the regions of maximum of $|\det(J(\mathbf{x}))|$.
- The dashed lines are the level sets of $|\det(J(\mathbf{x}))|$. The local error analysis does not take count of the global aspects of localization.