MICROFLUIDICS AND RAREFIED GAS DYNAMICS

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The Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{x}} = Q(f, f) \qquad (1)$$

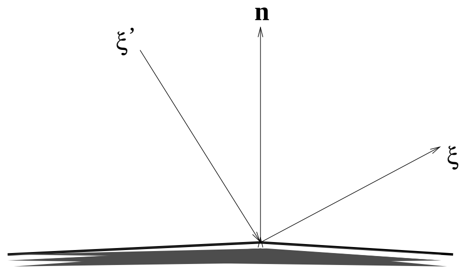
- $f(\mathbf{x}, \boldsymbol{\xi}, t)$ is the distribution function for the molecular velocity $\boldsymbol{\xi}$.

The collision integral is given by

$$Q(f, f) = \int_{R^3} d\boldsymbol{\xi}_* \int_0^{2\,\pi} d\epsilon \int_0^{\pi} d\Theta \, [f' \, f'_* - f \, f_*] \, B(\Theta, V) \qquad (2)$$

- $V = |\boldsymbol{\xi} - \boldsymbol{\xi}_*|$ is the relative velocity between two molecules
- f is a function of $\boldsymbol{\xi}$, while f_* refers to $\boldsymbol{\xi}_*$
- $f' \equiv f(\boldsymbol{\xi}')$, $f'_* \equiv f({\boldsymbol{\xi}'}_*)$, where $\boldsymbol{\xi}'$ and ${\boldsymbol{\xi}'}_*$ are the velocities after collision of two molecules with velocities $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_*$
- Θ is the angle through which the relative velocity has turned
- ϵ is the azimuthal angle the plane containing the relative velocities before and after collision makes with a fixed reference plane
- $B(\Theta, V)$ depends on the specific law of interaction between the molecules and is related to the differential scattering cross section.

Boundary conditions



$$f(\mathbf{x}, \boldsymbol{\xi}, t)|\boldsymbol{\xi} \cdot \hat{\mathbf{n}}| = \int_{\boldsymbol{\xi}' \cdot \hat{\mathbf{n}} < 0} R(\boldsymbol{\xi}' \to \boldsymbol{\xi};\, \mathbf{x}, t) f(\mathbf{x}, \boldsymbol{\xi}', t)|\boldsymbol{\xi}' \cdot \hat{\mathbf{n}}|\, d\boldsymbol{\xi}'$$

$$(\mathbf{x} \in \Omega,\ \boldsymbol{\xi} \cdot \hat{\mathbf{n}} > 0) \qquad (3)$$

- $R(\boldsymbol{\xi}' \to \boldsymbol{\xi};\, \mathbf{x}, t)$ is the scattering kernel.

Maxwell Model

$$R(\boldsymbol{\xi}' \to \boldsymbol{\xi}) = \alpha M_w(\boldsymbol{\xi})|\boldsymbol{\xi} \cdot \hat{\mathbf{n}}| + (1 - \alpha)\delta(\boldsymbol{\xi} - \boldsymbol{\xi}' + 2\,\hat{\mathbf{n}}(\boldsymbol{\xi}' \cdot \hat{\mathbf{n}}))$$

$$(\boldsymbol{\xi} \cdot \hat{\mathbf{n}} > 0;\ \boldsymbol{\xi}' \cdot \hat{\mathbf{n}} < 0) \qquad (4)$$

- $M_w(\boldsymbol{\xi})$ is the Maxwellian distribution of the boundary
- α is the accommodation coefficient $\Longrightarrow \alpha = 0$ (specular reflection), $\alpha = 1$ (diffuse reflection).

The linearized collision operator

Linearized problem

$$f = f_0(1 + h) \tag{5}$$

- f_0 is a Maxwellian distribution (usually with zero bulk velocity)

$$f_0 = \rho_0 (2\pi R T_0)^{-3/2} e^{-\dfrac{\xi^2}{(2RT_0)}} \tag{6}$$

where: ρ_0 and T_0 are the equilibrium density and temperature, respectively, and R is the gas constant.

- h is the perturbation upon the basic equilibrium state.

Inserting Eq. (5) in (2), the linearized collision operator can be written as follows

$$Lh = 2\, {f_0}^{-1} Q(f_0 h, f_0) \tag{7}$$

In order to obtain Eq. (7), we used the following properties of the collision integral

- $Q(f, g) = Q(g, f)$
- $Q(f_0, f_0) = 0$ if f_0 is a Maxwellian

A more explicit expression of Lh reads

$$Lh = \int_{R^3} d\boldsymbol{\xi}_* \int_0^{2\,\pi} d\epsilon \int_0^{\pi} d\Theta \; f_0(\boldsymbol{\xi}_*) \left[h' + h'_* - h - h_*\right] B(\Theta, V) \tag{8}$$

- h is a function of $\boldsymbol{\xi}$, while h_* refers to $\boldsymbol{\xi}_*$
- $h' \equiv h(\boldsymbol{\xi}')$, $h'_* \equiv h(\boldsymbol{\xi}'_*)$

The linearized boundary conditions

According to the scattering kernel theory, the boundary conditions for the perturbation h turn out to be

$$h^+ = h_0 + Kh^- \tag{9}$$

where h^+ and h^- concern, respectively, the reemitted and the impinging molecules. Eq. (9) is obtained by inserting Eq. (5) in the definition (3). In Eq. (9), the boundary source term, h_0, is given by

$$h_0 = [f_0(\boldsymbol{\xi})|\boldsymbol{\xi}\cdot\hat{\mathbf{n}}|]^{-1}\int_{\boldsymbol{\xi}'\cdot\hat{\mathbf{n}}<0} R(\boldsymbol{\xi}'\to\boldsymbol{\xi})f_0(\boldsymbol{\xi}')|\boldsymbol{\xi}'\cdot\hat{\mathbf{n}}|d\boldsymbol{\xi}' - 1 \qquad (\boldsymbol{\xi}\cdot\hat{\mathbf{n}}>0) \tag{10}$$

and K denotes the following operator

$$Kh^- = [f_0(\boldsymbol{\xi})|\boldsymbol{\xi}\cdot\hat{\mathbf{n}}|]^{-1}\int_{\boldsymbol{\xi}'\cdot\hat{\mathbf{n}}<0} R(\boldsymbol{\xi}'\to\boldsymbol{\xi})f_0(\boldsymbol{\xi}')|\boldsymbol{\xi}'\cdot\hat{\mathbf{n}}|h^-(\boldsymbol{\xi}')d\boldsymbol{\xi}' \qquad (\boldsymbol{\xi}\cdot\hat{\mathbf{n}}>0) \tag{11}$$

Kinetic models

When one is not interested in fine details, it is possible to obtain reasonable results by replacing the collision integral, $Q(f, f)$, by a so-called collision model, a simpler expression $J(f)$ that retains only the qualitative and average properties of the collision term.

The equation for the distribution function is then called a kinetic model or a model equation.

The most widely known collision model is usually called the Bhatnagar, Gross and Krook (BGK) model.

It reads as follows

$$J(f) = \nu[\Phi(\boldsymbol{\xi}) - f(\boldsymbol{\xi})] \tag{12}$$

- ν is the collision frequency independent of $\boldsymbol{\xi}$
- Φ denotes the local Maxwellian, that is, the (unique) Maxwellian having the same density, bulk velocity and temperature as f.

An advantage of the BGK model is offered by its linearized form

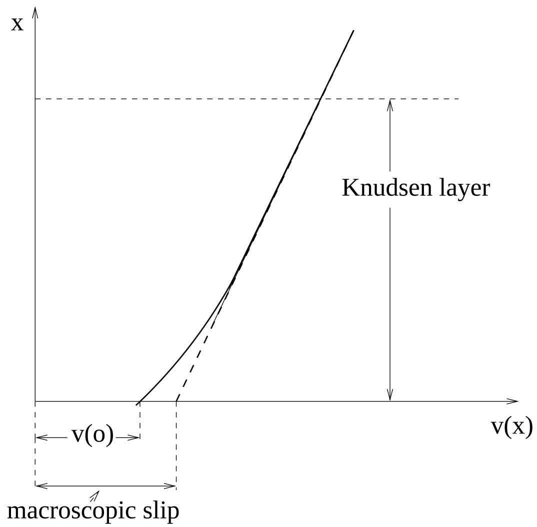
$$Lh^{BGK} = \nu_0 \Bigg[\int \hat{f}_0(\tilde{\boldsymbol{\xi}}) h(\tilde{\boldsymbol{\xi}}) d\tilde{\boldsymbol{\xi}} + \frac{\boldsymbol{\xi}}{RT_0} \cdot \int \tilde{\boldsymbol{\xi}} \hat{f}_0(\tilde{\boldsymbol{\xi}}) h(\tilde{\boldsymbol{\xi}}) d\tilde{\boldsymbol{\xi}}$$

$$+ \frac{2}{3} \left(\frac{|\boldsymbol{\xi}|^2}{2RT_0} - \frac{3}{2} \right) \int \left(\frac{|\tilde{\boldsymbol{\xi}}|^2}{2RT_0} - \frac{3}{2} \right) \hat{f}_0(\tilde{\boldsymbol{\xi}}) h(\tilde{\boldsymbol{\xi}}) d\tilde{\boldsymbol{\xi}} - h \Bigg] \qquad (13)$$

- $\hat{f}_0(\tilde{\boldsymbol{\xi}}) = \dfrac{f_0(\boldsymbol{\xi})}{\rho_0}$
- ν_0 is the collision frequency evaluated at the density ρ_0 and temperature T_0 of the unperturbed state.

The BGK linearized operator can be obtained by inserting Eq. (5) in Eq. (12) and by approximating the local Maxwellian Φ in terms of f_0 and the moments of h.

Knudsen layers

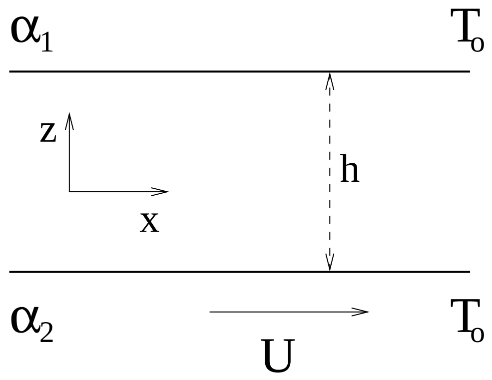


Microscopic slip: velocity of the gas at the wall $v(0)$.

Macroscopic slip: help us to reconstruct the correct behavior of the bulk velocity outside the Knudsen layer.

Because of the half-range nature of the boundary conditions, something unusual must happen near a wall: The molecules arriving there 'do not know' that there is a wall and have a distribution function that reflects the presence of a boundary only indirectly (because of the collisions they suffer with the molecules coming from the wall). It is clear then that, near a wall, there must be a layer, of the order of a few mean free paths, where the solution is widely different from that prevailing in the remaining part of the slab. Layers of this kind are called Knudsen layers or kinetic boundary layers .

The Poiseuille-Couette problem with gas-rarefaction effects



Steady-state Boltzmann equation:

$$c_x \frac{\partial f}{\partial x} + c_z \frac{\partial f}{\partial z} = Q(f, f)$$

Small pressure gradient, small $U \Longrightarrow$ linearized problem:

$$f = f_0(1 + \tilde{h}) \tag{14}$$

- $f(x, z, \mathbf{c})$ is the distribution function for the molecular velocity $\mathbf{c}$ expressed in units of $(2RT_0)^{1/2}$
- f_0 is the Maxwellian in equilibrium with the walls: $f_0(x, \mathbf{c}) = (1 + kx)\rho_0 \pi^{-3/2} e^{-c^2}$, where ρ_0 and T_0 are the equilibrium density and temperature, respectively, and R is the gas constant

- $k = \frac{1}{p}\frac{\partial p}{\partial x}$ is the pressure gradient
- $\tilde{h}(z, \mathbf{c})$ is the perturbation

Using Eq. (14), the linearized Boltzmann equation reads

$$k c_x + c_z \frac{\partial \tilde{h}}{\partial z} = L\tilde{h} \tag{15}$$

Linearized BGK model for molecular collisions:

$$\begin{aligned} L\tilde{h} = \quad & (\pi^{-3/2}/\theta)\bigg[\int e^{-c_1^2}\tilde{h}_1 d\mathbf{c}_1 + 2\mathbf{c}\cdot\int \mathbf{c}_1 e^{-c_1^2}\tilde{h}_1 d\mathbf{c}_1 \\ & + \tfrac{2}{3}(c^2 - \tfrac{3}{2})\int (c_1^2 - 3/2)e^{-c_1^2}\tilde{h}_1 d\mathbf{c}_1\bigg] - \tilde{h}/\theta \end{aligned} \tag{16}$$

where $\tilde{h}_1 \equiv \tilde{h}(z, \mathbf{c}_1)$ and θ is the collision time given by

$$\theta = \eta \frac{\sqrt{2RT_0}}{p}$$

with η being the gas viscosity.

Multiplying Eq. (15) by $(c_x/\pi)\exp[-(c_x^2+c_y^2)]$ and integrating with respect to c_x and c_y, it turns out:

$$\frac{1}{2}k + c_z\frac{\partial Z}{\partial z} = \frac{1}{\theta}\left[\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{-c_{z1}^2}Z(z,c_{z_1})\,dc_{z_1} - Z(z,c_z)\right] \tag{17}$$

where by definition:

$$Z(z,c_z) = \pi^{-1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-c_x^2-c_y^2}c_x\tilde{h}(z,\mathbf{c})\,dc_x\,dc_y \tag{18}$$

Hence, the integral equation for the bulk velocity of the gas can be written as follows:

$$q(z) = \pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{-c_{z1}^2}Z(z,c_{z_1})\,dc_{z_1} \tag{19}$$

If one assumes that $q(z)$ is a known quantity, the integrodifferential Boltzmann equation (17) can be formally handled as an ordinary inhomogeneous differential equation whose solution reads as

$$Z(z,c_z) = \exp\left[-\left(z+\frac{h}{2}\text{sgn}c_z\right)/(c_z\theta)\right] Z(-\tfrac{h}{2}\text{sgn}c_z, c_z) + \int_{-\frac{h}{2}sgnc_z}^{z} \exp\left(\frac{-|z-t|}{|c_z|\theta}\right)[q(t)-k\theta/2]/(c_z\theta)\,dt \tag{20}$$

with the values at the boundary $Z(-(h/2)\text{sgn}c_z, c_z)$ depending on the model of boundary conditions chosen.

Maxwell boundary conditions:

$$Z^+(h/2, c_z) = (1-\alpha_1) Z^-(h/2, -c_z)$$

$$Z^+(-h/2, c_z) = \alpha_2 U + (1-\alpha_2) Z^-(-h/2, -c_z)$$

- $Z^-(-h/2, c_z)$, $Z^-(h/2, c_z)$ are the distribution functions of the molecules impinging upon the walls
- $Z^+(-h/2, c_z)$, $Z^+(h/2, c_z)$ are the distribution functions of the molecules reemerging from the walls

Inserting in the definition (19) of $q(z)$ the Z function (20), the bulk velocity of the gas can be rewritten in terms of the nondimensional functions $\psi_p(u)$ and $\psi_c(u)$:

$$q(z) = \frac{1}{2}k\theta[1 - \psi_p(u)] + U\psi_c(u) \tag{21}$$

Poiseuille Flow

$$\begin{aligned}\psi_p(u) = \quad & 1 + \frac{1}{\sqrt{\pi}}\int_{-\delta/2}^{\delta/2} dw\psi_p(w)\Big\{(1-\alpha_1)S_{-1}(\delta - u - w) + (1-\alpha_2)S_{-1}(\delta + u + w) + \\ & (1-\alpha_1)(1-\alpha_2)[S_{-1}(2\delta - u + w) + S_{-1}(2\delta + u - w)] + T_{-1}(|u-w|)\Big\}\end{aligned}$$

Couette Flow

$$\begin{aligned}\psi_c(u) = \quad & \frac{\alpha_2}{\sqrt{\pi}}\Big[T_o(\delta/2 + u) + (1-\alpha_1)S_o(3/2\delta - u) + (1-\alpha_1)(1-\alpha_2)S_o(5/2\delta + u)\Big] + \\ & \frac{1}{\sqrt{\pi}}\int_{-\delta/2}^{\delta/2} dw\psi_c(w)\Big\{(1-\alpha_1)S_{-1}(\delta - u - w) + (1-\alpha_2)S_{-1}(\delta + u + w) + \\ & (1-\alpha_1)(1-\alpha_2)[S_{-1}(2\delta - u + w) + S_{-1}(2\delta + u - w)] + T_{-1}(|u-w|)\Big\}\end{aligned}$$

- $T_n(x)$ is the Abramowitz function defined by

$$T_n(x) = \int_0^{+\infty} t^n \exp(-t^2 - x/t)\, dt$$

- $S_n(x)$ is a generalized Abramowitz function defined by

$$S_n(x, \delta, \alpha_1, \alpha_2) = \int_0^{+\infty} \frac{t^n \exp(-t^2 - x/t)}{1 - (1 - \alpha_1)(1 - \alpha_2) \exp(-2\delta/t)}\, dt$$

- non-dimensional variables

$$\delta = h/\theta, \qquad w = t/\theta, \qquad u = z/\theta$$

where δ is the rarefaction parameter given by $\delta = \sqrt{\pi}/(2Kn)$, with $Kn = \lambda/h$ being the Knudsen number (λ is the mean free path of the gas molecules).

The flow rate (per unit time through unit thickness) defined by

$$F = \rho \int_{-h/2}^{h/2} q(z) dz$$

can be expressed as the sum of the Poiseuille flow (F_p) and the Couette flow (F_c) as follows

$$F = F_p + F_c = -\frac{\partial p}{\partial x} h^2 Q_p(\delta, \alpha_1, \alpha_2) + \frac{\rho U h}{2} Q_c(\delta, \alpha_1, \alpha_2) \tag{22}$$

where

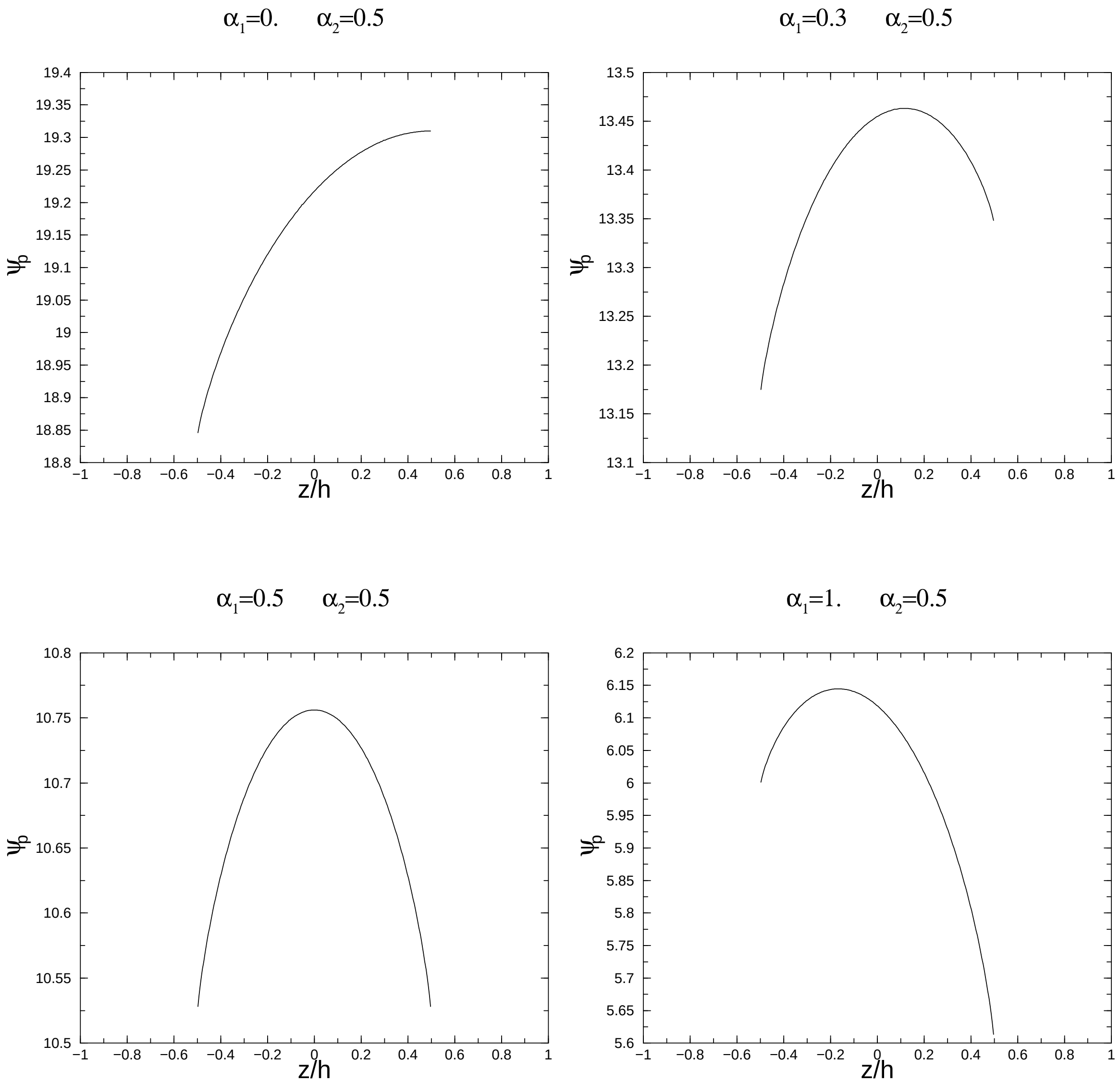
$$Q_p(\delta, \alpha_1, \alpha_2) = -\frac{1}{\delta} + \frac{1}{\delta^2} \int_{-\delta/2}^{\delta/2} \psi_p(u) du$$

$$Q_c(\delta, \alpha_1, \alpha_2) = \frac{2}{\delta} \int_{-\delta/2}^{\delta/2} \psi_c(u) du$$

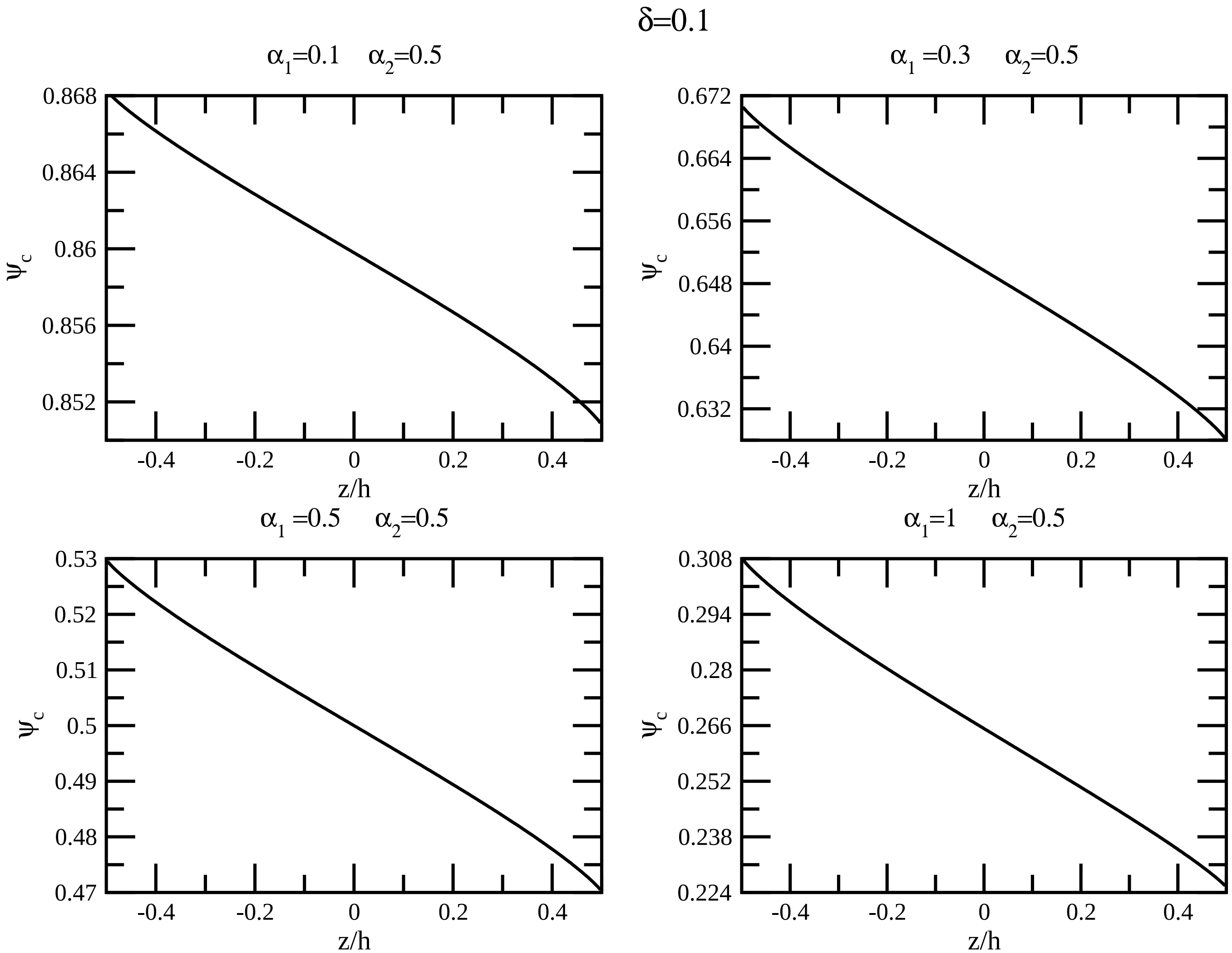
are the non-dimensional volume flow rates.

Poiseuille velocity profiles

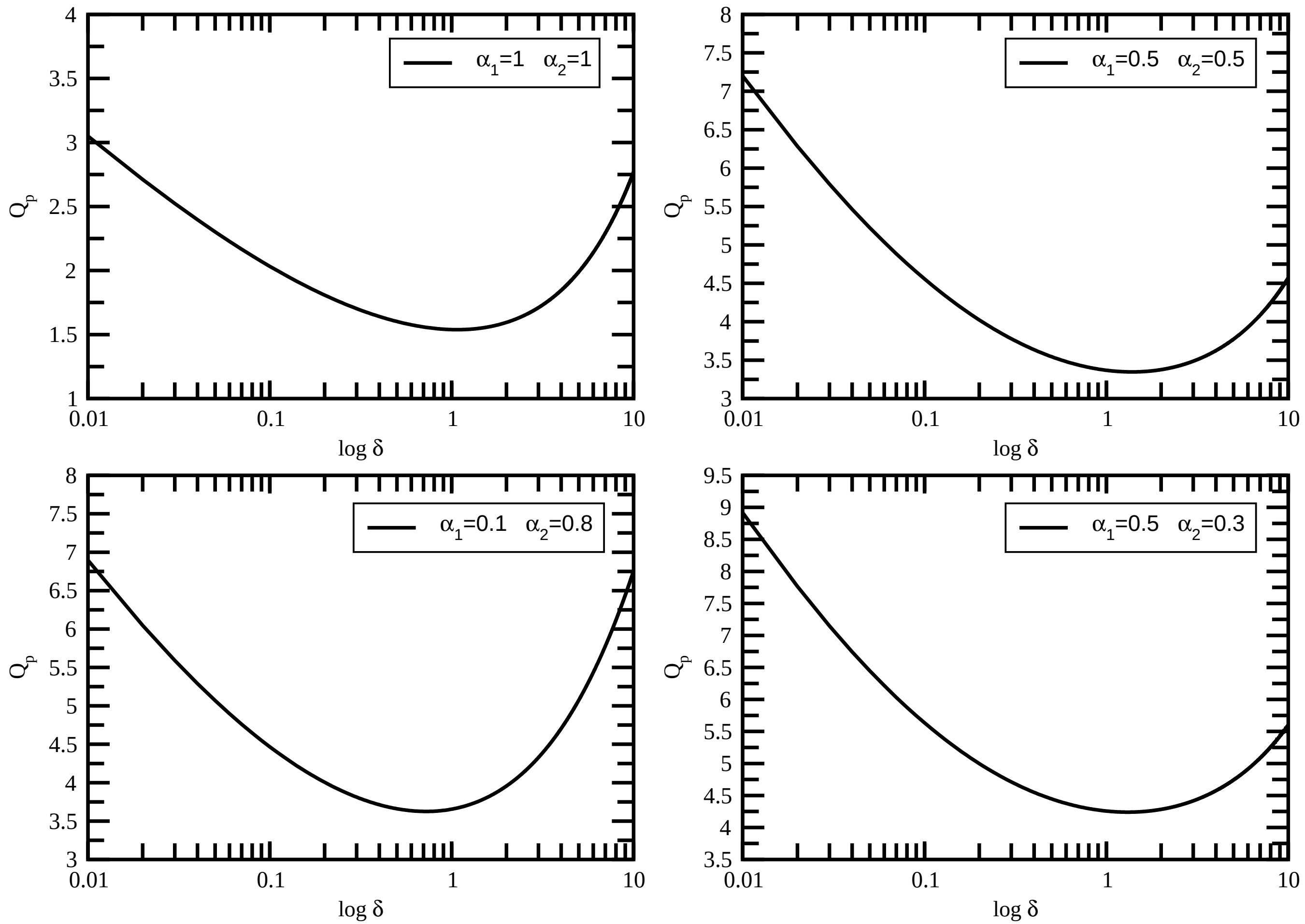
$$\delta = 10^{-3}$$



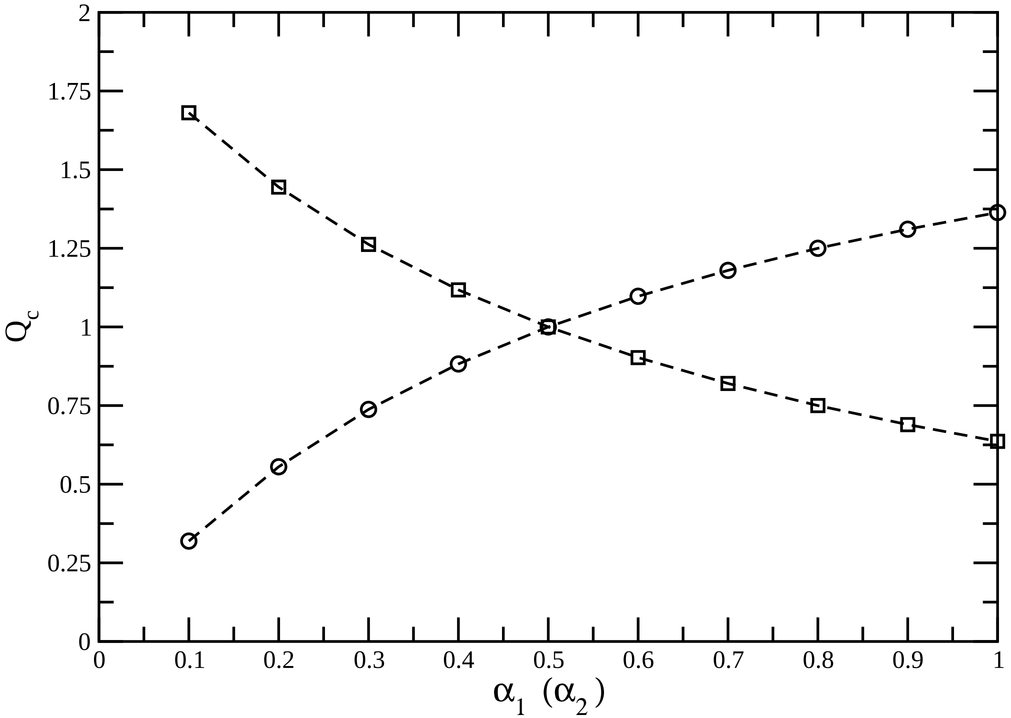
Couette velocity profiles



Poiseuille flow rate



Couette flow rate



- The squares are obtained varying α_1 ($\alpha_2 = 0.5$)
- The circles are obtained varying α_2 ($\alpha_1 = 0.5$)

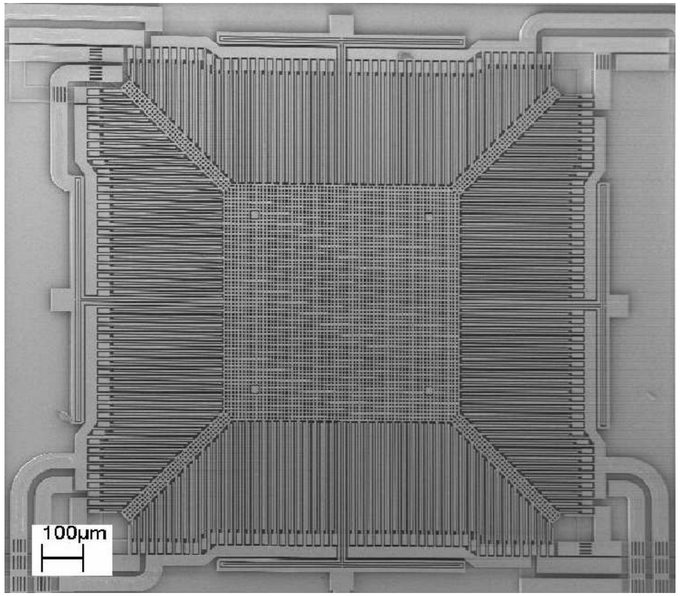
Squeezed-film dampers with low oscillation frequency

Micromechanical accelerometers, characterized by very small gaps between the moving elements and the fixed electrodes, often use a gas as damping medium.

The damping, due to the internal friction of the flowing gas, in the small gaps between these oscillating microstructures, is an important design parameter since it determines, e.g., the frequency-domain behavior of the sensor or the quality factor of the vibrating filter structure.

At low pressures or in ultra thin films, the gas rarefaction effects and the molecular interaction with the surfaces effectively change the viscosity. Therefore, in this flow regime, the continuum equations are no longer valid and the Boltzmann equation must be considered.

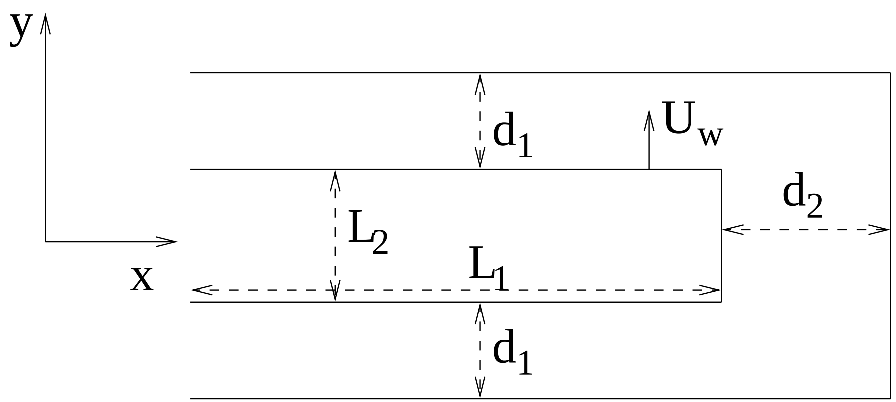
Silicon biaxial accelerometer



In spite of its apparently complex structure, a real micromechanical accelerometer usually has a highly repetitive layout whose basic units consist of two or three-dimensional microchannels where different sets of bounding walls move in the direction perpendicular or parallel to their surfaces.

Biaxial accelerometer: single unit

Two-dimensional section



- $\mathrm{d}_1 = 2.6 \cdot 10^{-6}$ m; $\mathrm{d}_2 = 4.2 \cdot 10^{-6}$ m; $\mathrm{L}_1 = 15 \cdot 10^{-6}$ m; $\mathrm{L}_2 = 3.9 \cdot 10^{-6}$ m.
- Air viscosity $\Longrightarrow \eta = 1.8 \cdot 10^{-5}\ \mathrm{Nsm}^{-2}$.

Linearized BGK Model for the Boltzmann Equation

Very low Mach number, quasi-static flow field $\Longrightarrow$ linearized problem:

$$f = f_0(1+h)$$

- $f(\mathbf{x},\mathbf{c})$ is the distribution function for $\mathbf{c}$ expressed in units of $(2RT_0)^{1/2}$
- f_0 is the absolute Maxwellian: $f_0(\mathbf{c}) = \rho_0 \pi^{-3/2} e^{-c^2}$
- $h(\mathbf{x},\mathbf{c})$ is the small perturbation
- ρ_0 and T_0 are the equilibrium density and temperature, respectively.

Linearized BGK model for molecular collisions $\Longrightarrow$ the Boltzmann equation reads:

$$\begin{aligned} c_x \frac{\partial h}{\partial \tilde{x}} + c_y \frac{\partial h}{\partial \tilde{y}} = \quad & \pi^{-3/2} \Big[\int e^{-c'^2} h(\tilde{x},\tilde{y},\mathbf{c}') d\mathbf{c}' + 2c_x \int c'_x e^{-c'^2} h(\tilde{x},\tilde{y},\mathbf{c}') d\mathbf{c}' + \\ & 2c_y \int c'_y e^{-c'^2} h(\tilde{x},\tilde{y},\mathbf{c}') d\mathbf{c}' \Big] - h(\tilde{x},\tilde{y},\mathbf{c}) \end{aligned} \tag{23}$$

- $\tilde{x} = x/\theta, \ \tilde{y} = y/\theta,$ with θ being the collision time.

Multiplying Eq. (23) by $(1/\sqrt{\pi})\exp(-c_z^2)$ and integrating with respect to c_z:

$$
\begin{aligned}
c_x \frac{\partial}{\partial \tilde{x}} \mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) + \quad c_y \frac{\partial}{\partial \tilde{y}} \mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) &= -\mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) + \\
&\rho(\tilde{x}, \tilde{y}) + 2c_x v_x(\tilde{x}, \tilde{y}) + 2c_y v_y(\tilde{x}, \tilde{y})
\end{aligned}
\tag{24}
$$

- $\mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y)$ is the reduced distribution function

$$
\mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-c_z^2} h(\tilde{x}, \tilde{y}, \mathbf{c}) dc_z
$$

- $\rho(\tilde{x}, \tilde{y})$ is the perturbation part of the density of molecules

$$
\rho(\tilde{x}, \tilde{y}) = \pi^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(c_x^2 + c_y^2)} \mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) \, dc_x dc_y
$$

- $v_x(\tilde{x}, \tilde{y})$ is the x-component of the bulk velocity of the gas

$$
v_x(\tilde{x}, \tilde{y}) = \pi^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} c_x e^{-(c_x^2 + c_y^2)} \mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) \, dc_x dc_y
$$

- $v_y(\tilde{x}, \tilde{y})$ is the y-component of the bulk velocity of the gas

$$
v_y(\tilde{x}, \tilde{y}) = \pi^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} c_y e^{-(c_x^2 + c_y^2)} \mathcal{H}(\tilde{x}, \tilde{y}, c_x, c_y) \, dc_x dc_y
$$

Linearized boundary conditions: Maxwell diffusion

$$\mathcal{H}(\tilde{x}, \tilde{y} = -\tilde{L}_2/2 - \delta_1, c_x, c_y) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dc'_x \int_{-\infty}^{0} dc'_y c'_y e^{-({c'}_x^2 + {c'}_y^2)} \, . \, \mathcal{H}(\tilde{x}, \tilde{y} = -\tilde{L}_2/2 - \delta_1, c'_x, c'_y) \qquad c_y > 0 \tag{25}$$

$$\mathcal{H}(\tilde{x}, \tilde{y} = -\tilde{L}_2/2, c_x, c_y) = (-\sqrt{\pi} + 2c_y)U_w + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dc'_x \int_{0}^{+\infty} dc'_y c'_y e^{-({c'}_x^2 + {c'}_y^2)} \, . \, \mathcal{H}(\tilde{x}, \tilde{y} = -\tilde{L}_2/2, c'_x, c'_y) \qquad c_y < 0 \tag{26}$$

$$\mathcal{H}(\tilde{x}, \tilde{y} = \tilde{L}_2/2, c_x, c_y) = (\sqrt{\pi} + 2c_y)U_w - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dc'_x \int_{-\infty}^{0} dc'_y c'_y e^{-({c'}_x^2 + {c'}_y^2)} \, . \, \mathcal{H}(\tilde{x}, \tilde{y} = \tilde{L}_2/2, c'_x, c'_y) \qquad c_y > 0 \tag{27}$$

$$\mathcal{H}(\tilde{x},\tilde{y}=\tilde{L}_2/2+\delta_1,c_x,c_y) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dc'_x \int_{0}^{+\infty} dc'_y c'_y e^{-(c'^2_x+c'^2_y)} \,. \\ \mathcal{H}(\tilde{x},\tilde{y}=\tilde{L}_2/2+\delta_1,c'_x,c'_y) \qquad c_y<0 \tag{28}$$

$$\mathcal{H}(\tilde{x}=\tilde{L}_1/2,\tilde{y},c_x,c_y) = 2c_yU_w - \frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dc'_y \int_{-\infty}^{0} dc'_x c'_x e^{-(c'^2_x+c'^2_y)} \,. \\ \mathcal{H}(\tilde{x}=\tilde{L}_1/2,\tilde{y},c'_x,c'_y) \qquad c_x>0 \tag{29}$$

$$\mathcal{H}(\tilde{x}=\tilde{L}_1/2+\delta_2,\tilde{y},c_x,c_y) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty} dc'_y \int_{0}^{+\infty} dc'_x c'_x e^{-(c'^2_x+c'^2_y)} \,. \\ \mathcal{H}(\tilde{x}=\tilde{L}_1/2+\delta_2,\tilde{y},c'_x,c'_y) \qquad c_x<0 \tag{30}$$

- $\tilde{L}_1 = L_1/\theta,\ \tilde{L}_2 = L_2/\theta,\ \delta_1 = d_1/\theta,\ \delta_2 = d_2/\theta$

In order to compute the forces exerted by the fluid on the rotor, the following elements of the stress tensor have been evaluated:

$$P_{xx}(\tilde{x},\tilde{y}) = \frac{\rho_0}{2} + \frac{\rho_0}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dc_x dc_y c_x^2 e^{-(c_x^2+c_y^2)}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) \tag{31}$$

$$P_{yy}(\tilde{x},\tilde{y}) = \frac{\rho_0}{2} + \frac{\rho_0}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dc_x dc_y c_y^2 e^{-(c_x^2+c_y^2)}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) \tag{32}$$

$$P_{xy}(\tilde{x},\tilde{y}) = \frac{\rho_0}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dc_x dc_y c_x c_y e^{-(c_x^2+c_y^2)}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) \tag{33}$$

Finite-Difference Method

Long-time behavior of the solution of the initial and boundary-value problem:

$$\frac{\partial}{\partial t}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) \quad + c_x\frac{\partial}{\partial \tilde{x}}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) + c_y\frac{\partial}{\partial \tilde{y}}\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) = -\mathcal{H}(\tilde{x},\tilde{y},c_x,c_y) + \rho(\tilde{x},\tilde{y}) + 2c_x v_x(\tilde{x},\tilde{y}) + 2c_y v_y(\tilde{x},\tilde{y}) \tag{34}$$

- Initial condition $\Longrightarrow \mathcal{H} = 0$
- Boundary conditions $\Longrightarrow$ Eqs. (25)-(30)

Deterministic finite-difference method $\Longrightarrow$ first order implicit scheme:

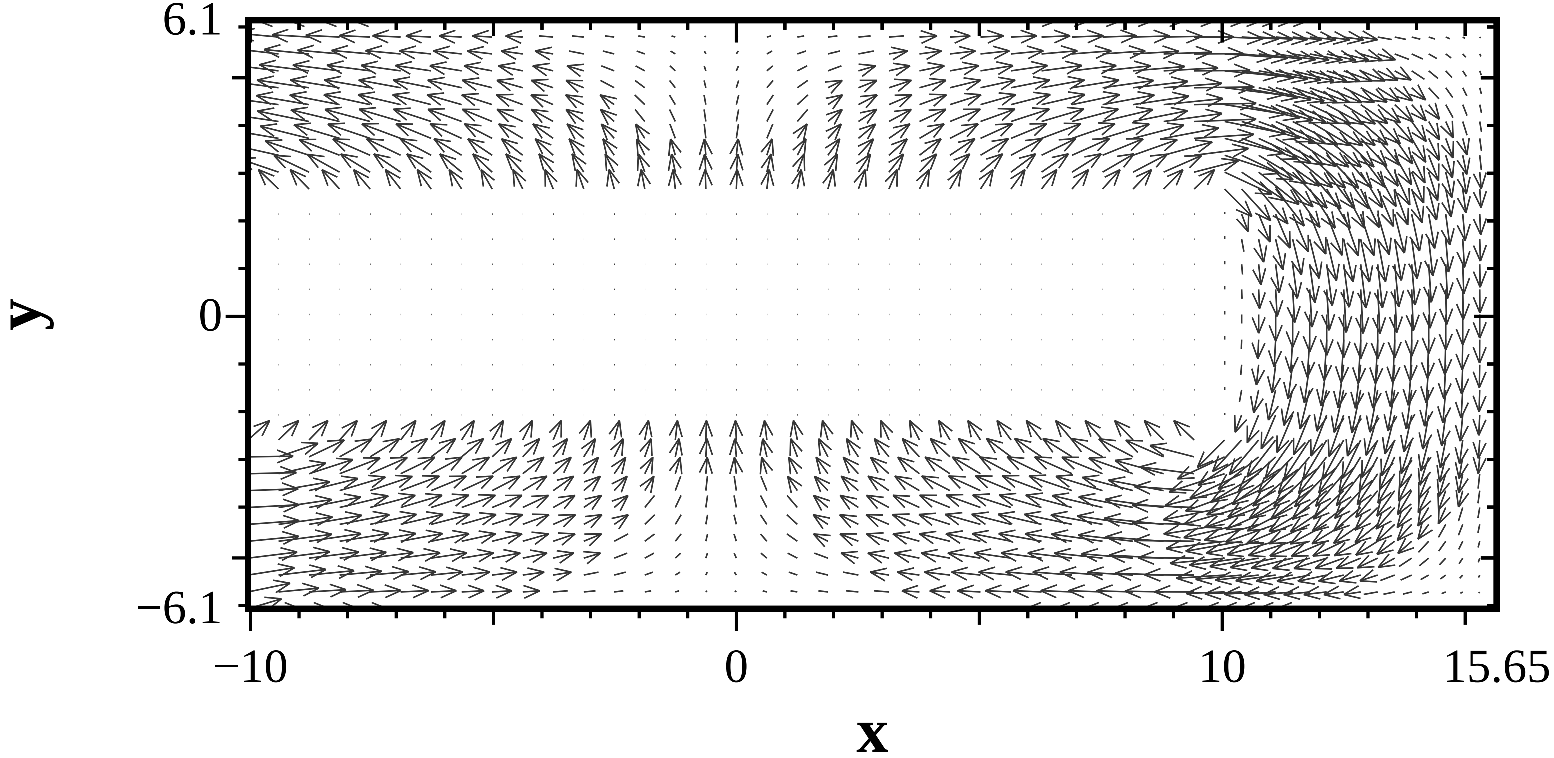
$$\mathcal{H}_{i,j}^{(n+1)}(l,m) \quad = [1 + r_i(l) + r_j(m) + \Delta t]^{-1} \times \{\mathcal{H}_{i,j}^{(n)}(l,m) + r_i(l)\mathcal{H}_{i-s,j}^{(n+1)}(l,m) + r_j(m)\mathcal{H}_{i,j-w}^{(n+1)}(l,m) + \Delta t[2c_x(l)v_{x_{i,j}}^{(n)} + 2c_y(m)v_{y_{i,j}}^{(n)} + \rho_{i,j}^{(n)}]\} \tag{35}$$

- $s = \text{sgn}(c_x(l))$, $w = \text{sgn}(c_y(m))$
- $r_i(l) = \dfrac{\Delta t|c_x(l)|}{\Delta x_i}$, $r_j(m) = \dfrac{\Delta t|c_y(m)|}{\Delta y_j}$

In Eq. (35) any quantity in the form $q_{i,j}^{(n)}(l,m)$ denotes the value of the corresponding function $q(x,y,t,c_x,c_y)$ evaluated in $x = x_i$, $y = y_j$, $t = n\Delta t$, $c_x = c_x(l)$, $c_y = c_y(m)$, being x_i and y_j the spatial coordinates of the center of the i-th cell whose width is Δx_i and the j-th cell whose width is Δy_j, respectively, Δt is the time step, $c_x(l)$ and $c_y(m)$ are the coordinates of the centers of the cell (l,m) in the velocity space.

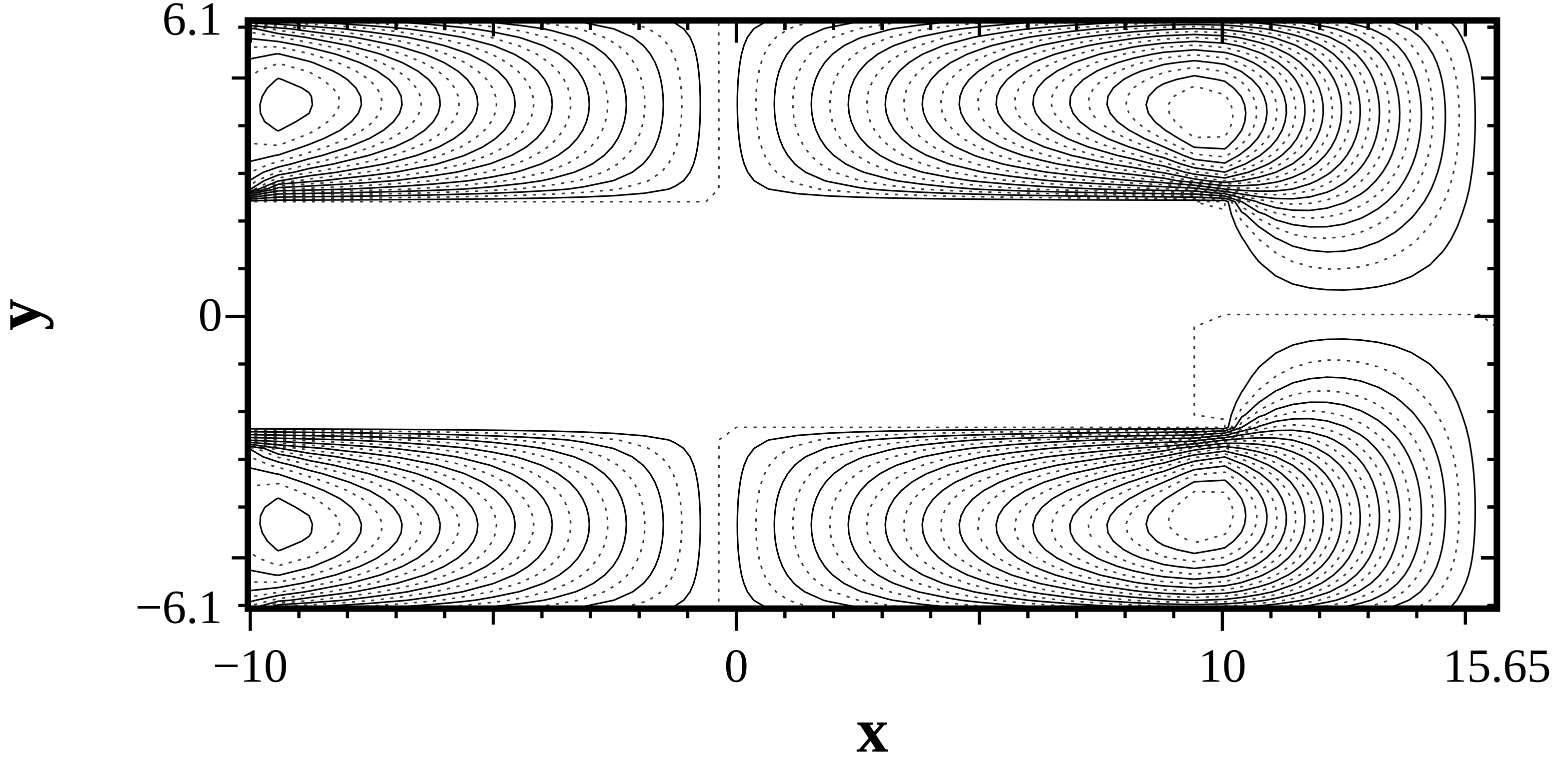
The grid parameters were chosen so as to provide the computational error of the bulk velocity and density fields within 1%.

Bulk velocity



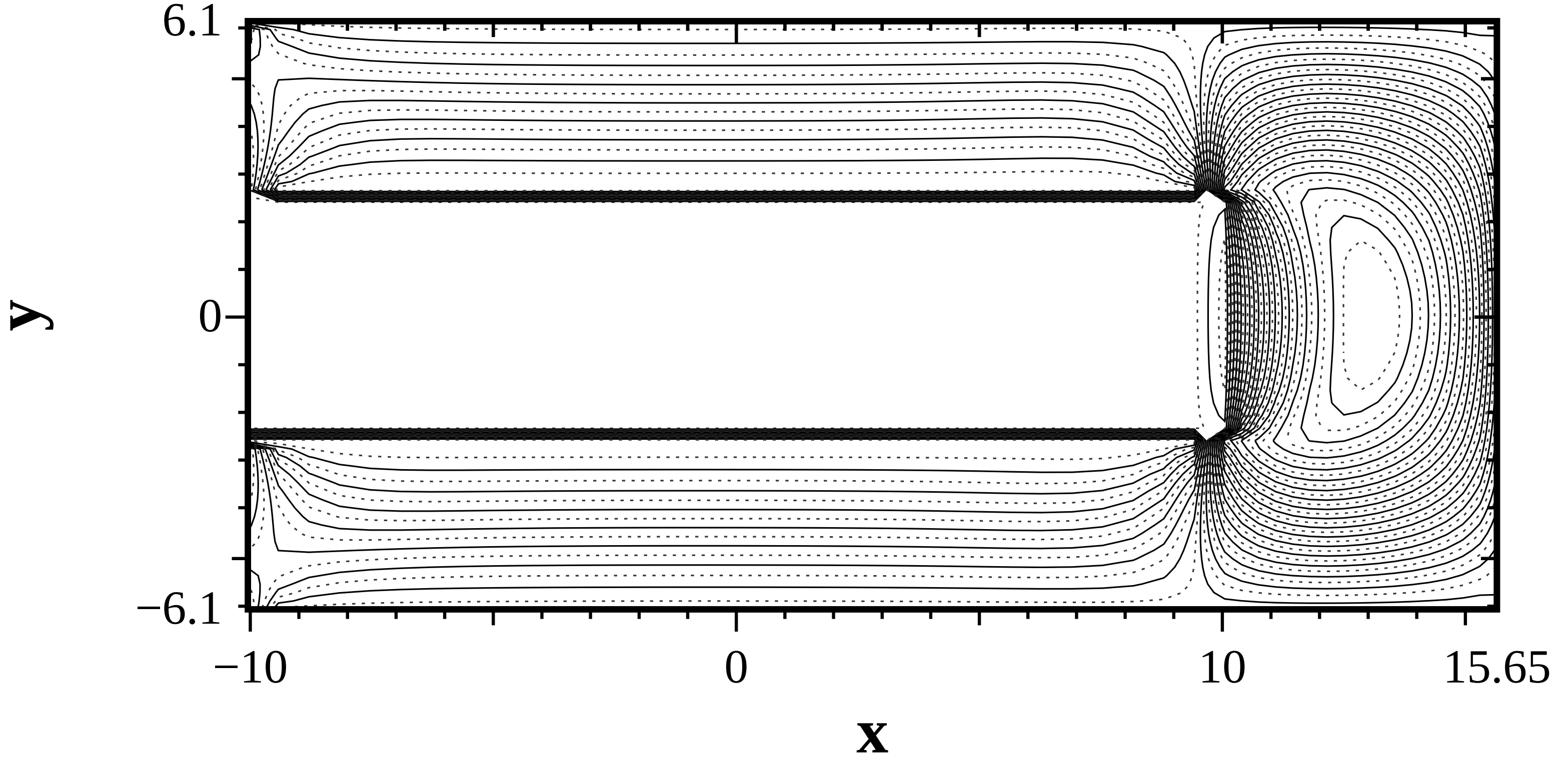
- $p = 0.1$ bar; $Kn \simeq 0.29$.

x-component of the bulk velocity: v_x



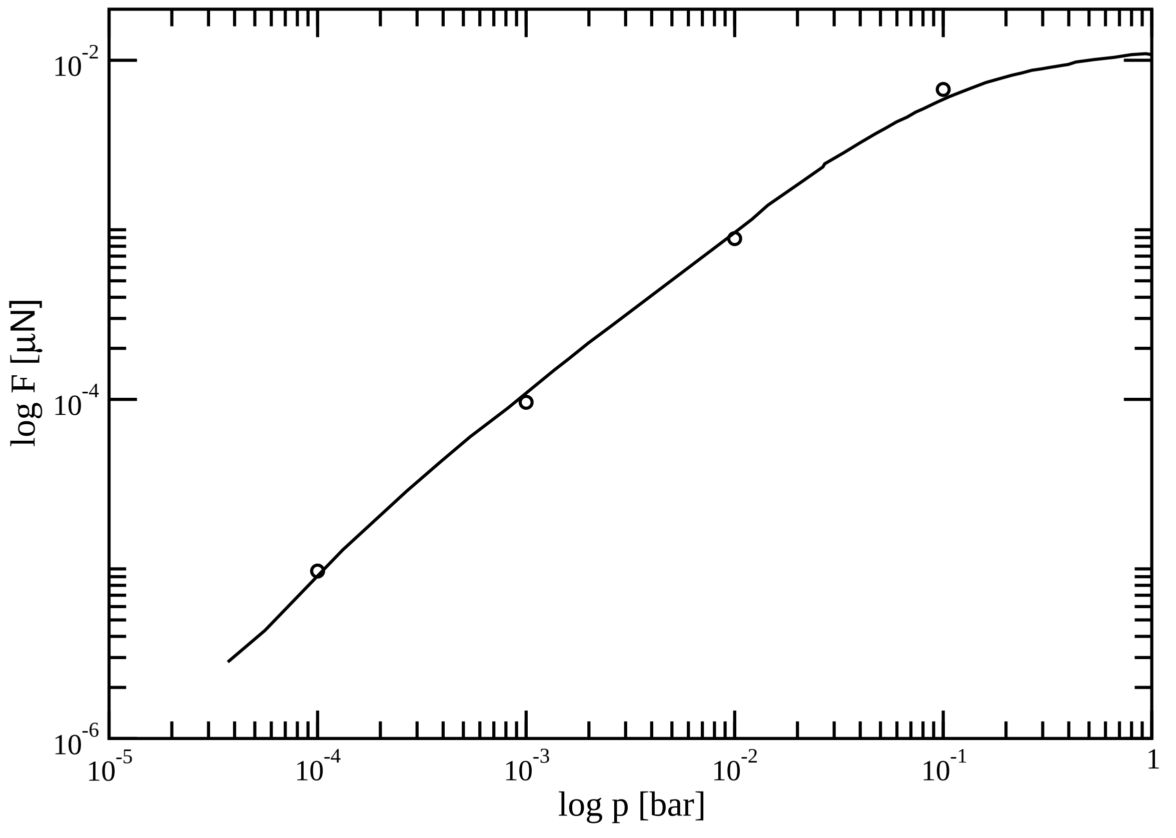
- $p = 0.1$ bar

y-component of the bulk velocity: v_y



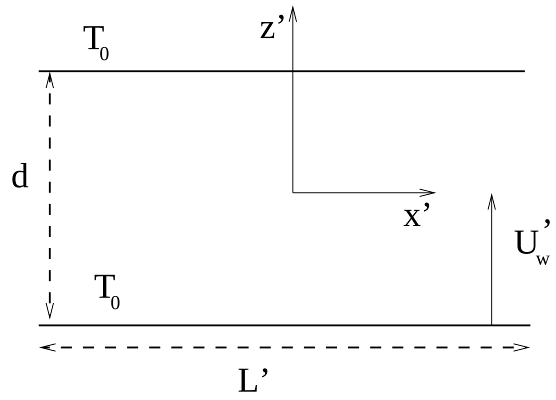
- $p = 0.1$ bar

Damping forces exerted by the fluid on the rotor



- Experimental data collected by STMicroelectronics (solid line)
- Numerical findings (circles)

Squeezed-film dampers with high oscillation frequency



High frequency devices (ranging from a few kHz to tens of MHz) find applications in inertial sensing, acoustic transduction, optical signal manipulation, and RF (radio frequency) components.

Besides the viscous forces that dominate at low frequencies, gas compressibility and inertial forces determine the amount of the damping force at higher oscillation frequencies.

We consider a monoatomic gas between two plane walls located at $z' = \pm d/2$.

Linearized BGK Model for the Boltzmann Equation

At $t' = 0$ the lower wall starts to oscillate perpendicularly to its own plane in the z'-direction with velocity

$$U'_w(t') = U'_0 \sin(\omega' t')$$

We assume that U'_0 is small enough

$$U'_0 << \sqrt{2RT_0} \implies \text{small Mach number}$$

so that the governing equations and the boundary conditions can be linearized about a Maxwellian f_0 by putting

$$f = f_0(1 + h)$$

- $f(x', z', \mathbf{c}, t')$ is the distribution function for $\mathbf{c}$ expressed in units of $(2RT_0)^{1/2}$
- f_0 is the absolute Maxwellian: $f_0(\mathbf{c}) = \rho_0 \pi^{-3/2} e^{-c^2}$
- $h(x', z', \mathbf{c}, t')$ is the small perturbation
- ρ_0 is the equilibrium density.

Rescaling all variables

$$t = \frac{t'}{\theta}, \qquad x = \frac{x'}{(2RT_0)\,\theta}, \qquad z = \frac{z'}{(2RT_0)\,\theta}$$

(with θ being the collision time)

the linearized BGK-Boltzmann equation reads:

$$\frac{\partial h}{\partial t} + c_x \frac{\partial h}{\partial x} + c_z \frac{\partial h}{\partial z} = \rho + 2\mathbf{c}\cdot\mathbf{v} + (\mathbf{c}^2 - 3/2)\,\tau - h \tag{36}$$

where

$$\rho = \pi^{-3/2} \int_{-\infty}^{+\infty} h\, e^{-\mathbf{c}^2}\, d\mathbf{c}, \tag{37}$$

$$\mathbf{v} = \pi^{-3/2} \int_{-\infty}^{+\infty} h\, \mathbf{c}\, e^{-\mathbf{c}^2}\, d\mathbf{c}, \tag{38}$$

$$\tau = \pi^{-3/2} \int_{-\infty}^{+\infty} h\, (\frac{2}{3}\mathbf{c}^2 - 1)\, e^{-\mathbf{c}^2}\, d\mathbf{c}, \tag{39}$$

First, we multiply Eq. (36) by $[1/\sqrt{\pi}\,\exp(-c_y^2)]$ and we integrate over all c_y; then, we multiply Eq. (36) by $[1/\sqrt{\pi}(c_y^2 - 1/2)\,\exp(-c_y^2)]$ and we integrate over all c_y (projection procedure):

$$\frac{\partial H}{\partial t} + c_x \frac{\partial H}{\partial x} + c_z \frac{\partial H}{\partial z} + H = \rho + 2\,c_x\,v_x + 2\,c_z\,v_z + (c_x^2 + c_z^2 - 1)\,\tau \tag{40}$$

$$\frac{\partial \Psi}{\partial t} + c_x \frac{\partial \Psi}{\partial x} + c_z \frac{\partial \Psi}{\partial z} + \Psi = \frac{\tau}{2} \tag{41}$$

where the reduced unknown distribution functions H and Ψ are defined as

$$H(x, z, c_x, c_z, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} h(x, z, \mathbf{c}, t)\, e^{-c_y^2}\, dc_y \tag{42}$$

$$\Psi(x, z, c_x, c_z, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} h(x, z, \mathbf{c}, t)\,(c_y^2 - 1/2)\, e^{-c_y^2}\, dc_y \tag{43}$$

In order to derive Eqs. (40) and (41), we have considered the linearized equation of state

$$P = \frac{1}{3}\left[P_{xx} + P_{yy} + P_{zz}\right] = \frac{1}{2}\left(\rho + \tau\right) \tag{44}$$

with P being the dimensionless perturbed pressure of the gas.

The macroscopic quantities appearing in the right-hand side of Eqs. (40) and (41) are defined by

$$\rho(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H\, e^{-(c_x^2+c_z^2)}\, dc_x dc_z \tag{45}$$

$$v_x(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H\, c_x\, e^{-(c_x^2+c_z^2)}\, dc_x dc_z \tag{46}$$

$$v_z(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H\, c_z\, e^{-(c_x^2+c_z^2)}\, dc_x dc_z \tag{47}$$

$$\tau(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{2}{3}\left[(c_x^2 + c_z^2 - 1)\, H + \Psi\right] e^{-(c_x^2+c_z^2)}\, dc_x dc_z \tag{48}$$

Linearized boundary conditions: Maxwell diffusion $(\alpha = 1)$

$$H(x, z = -\delta/2, c_x, c_z, t) = (\sqrt{\pi} + 2\, c_z)\, U_w$$
$$- \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\tilde{c}_x \int_{\tilde{c}_z < 0} d\tilde{c}_z\, \tilde{c}_z\, e^{-(\tilde{c}_x^2 + \tilde{c}_z^2)}\, H(x, z = -\delta/2, \tilde{c}_x, \tilde{c}_z, t) \quad c_z > 0$$

$$\Psi(x, z = -\delta/2, c_x, c_z, t) = 0 \qquad c_z > 0$$

$$H(x, z = \delta/2, c_x, c_z, t) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\tilde{c}_x \cdot$$
$$\int_{\tilde{c}_z > 0} d\tilde{c}_z\, \tilde{c}_z\, e^{-(\tilde{c}_x^2 + \tilde{c}_z^2)}\, H(x, z = \delta/2, c_x, c_z, t) \qquad c_z < 0$$

$$\Psi(x, z = \delta/2, c_x, c_z, t) = 0 \qquad c_z < 0$$

- $\delta = d/(\sqrt{2RT_0}\,\theta)$ is the rarefaction parameter (inverse Knudsen number)

- U_w is the dimensionless wall velocity given by $U_w(t) = U_0\,\sin(\omega\,t)$

with $U_w = U_w'/\sqrt{2RT_0}$, $U_0 = U_0'/\sqrt{2RT_0}$, $\omega = \theta\,\omega'$, $T = 2\pi/\omega = T'/\theta$.

In order to compute the forces exerted by the gas on the moving wall, the following elements of the stress tensor have to be evaluated at $z = -\delta/2$

$$P_{zz}(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H\,c_z^2\,e^{-(c_x^2+c_z^2)}\,dc_x dc_z \tag{49}$$

$$P_{xz}(x, z, t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H\,c_x\,c_z\,e^{-(c_x^2+c_z^2)}\,dc_x dc_z \tag{50}$$

The dominant contribution is given by the component P_{zz}.

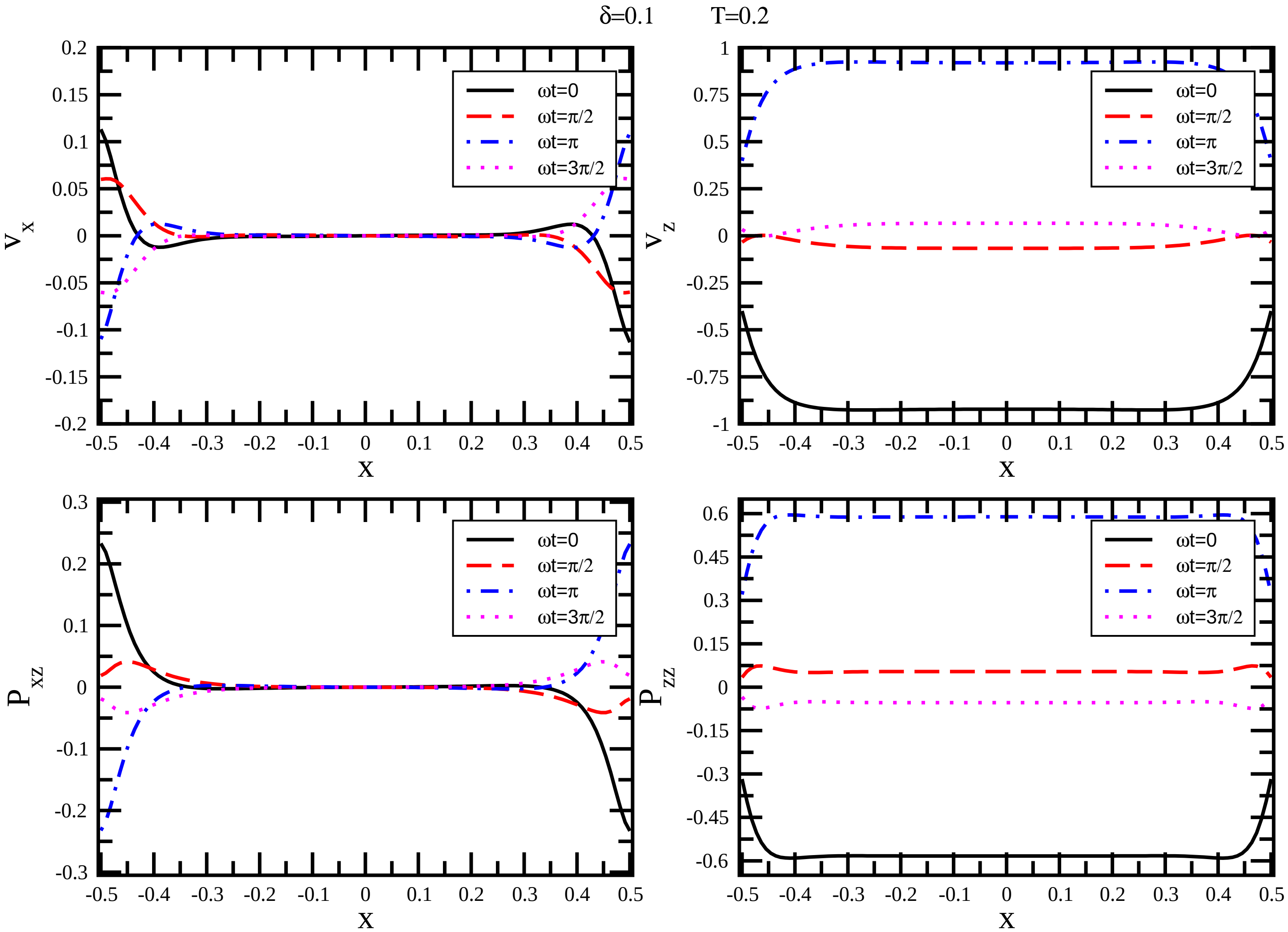
The numerical results show that above a certain frequency of oscillation, the sound wave propagation takes place only in the z-direction across the gap, which indicates a fully trapped gas situation.

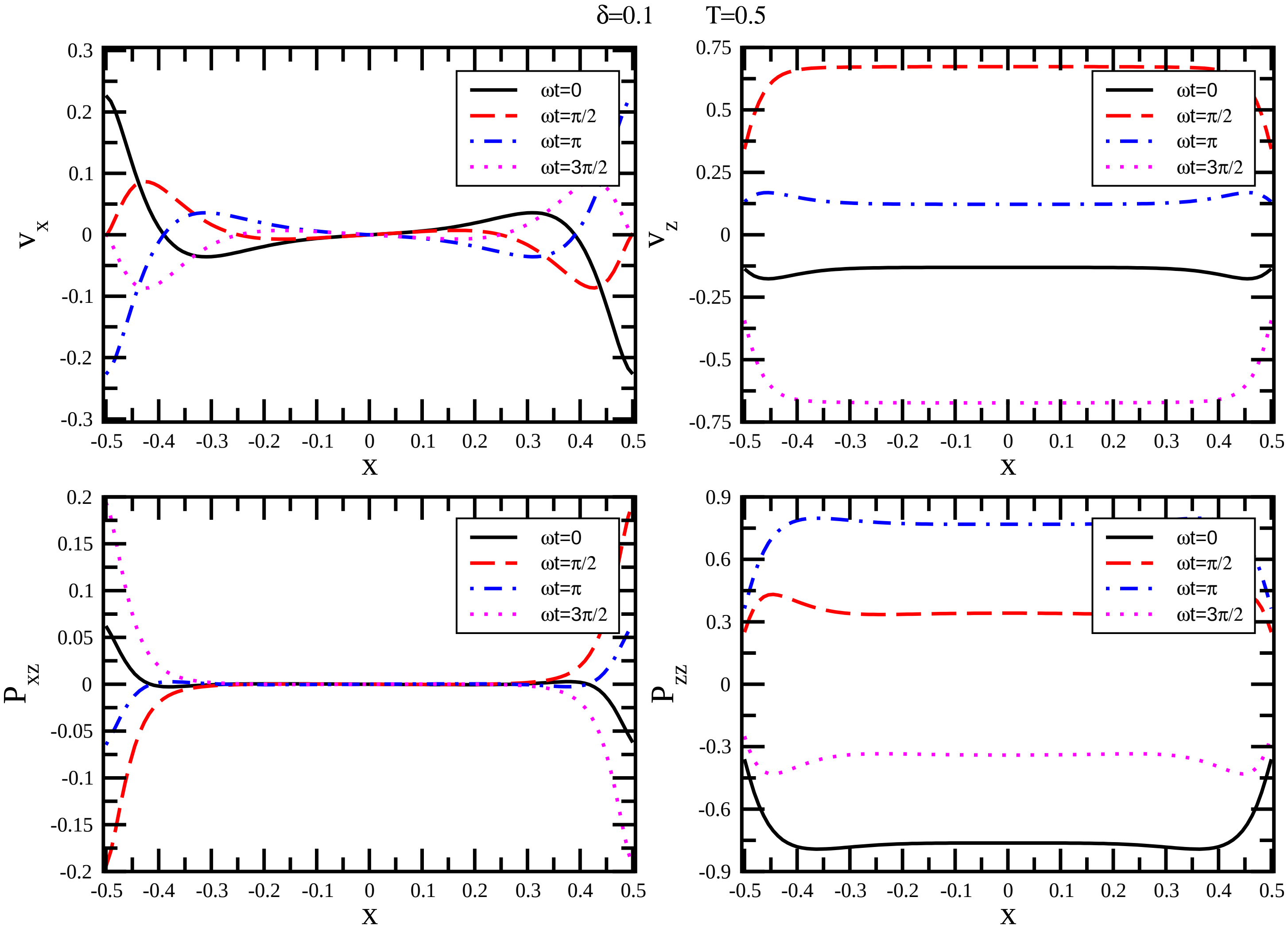
This assumption greatly simplifies the analysis since the topology of the damper becomes insignificant and the problem can be reduced to a 1-dimensional one.

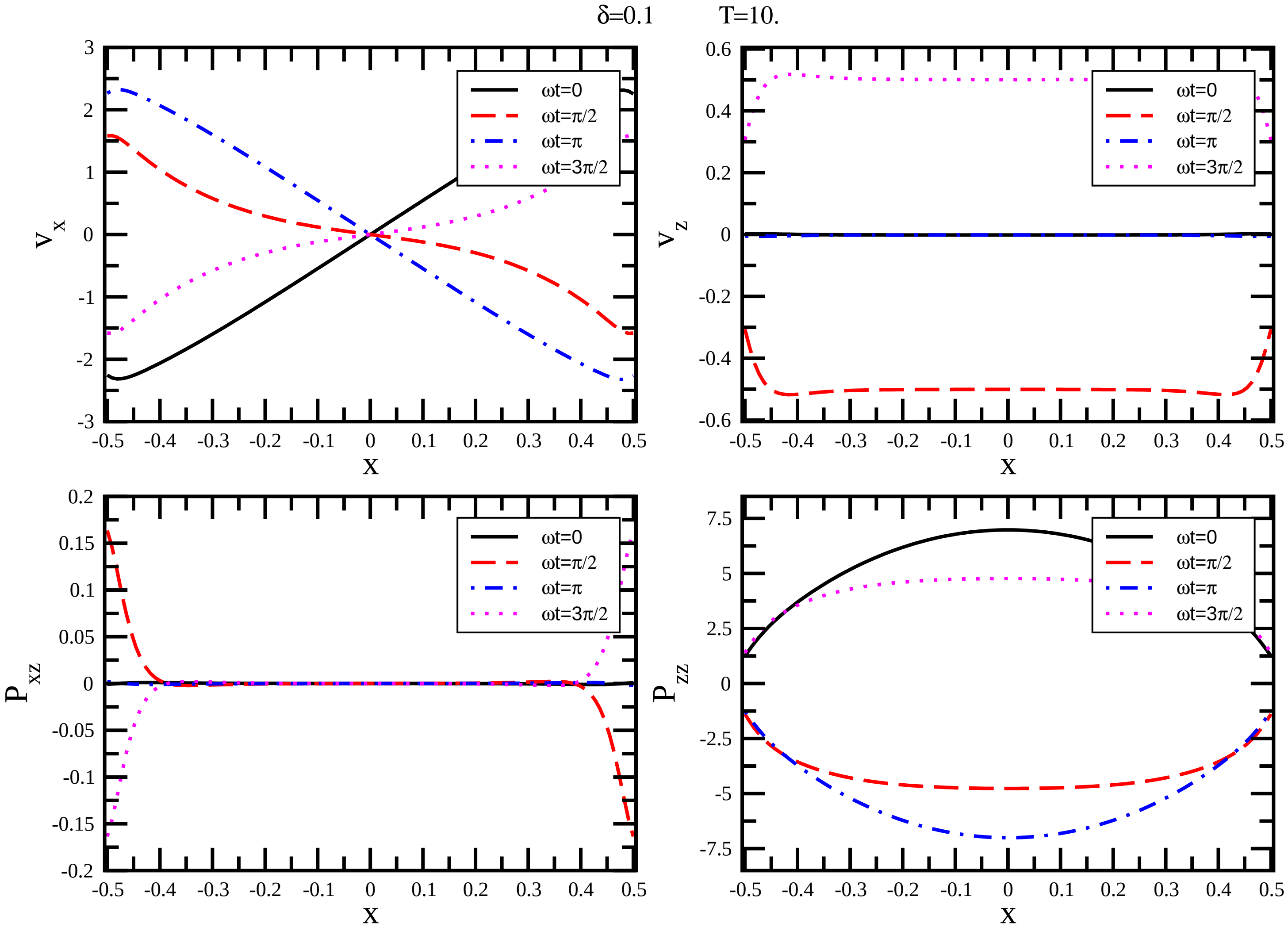
To demonstrate the gas trapping in the channel gap at high frequencies, the following Figures show the variations of the macroscopic fields of interest along the channel for different periods of oscillation.

The profiles reported in these pictures (at different stages during a period of oscillation after the transient behavior has ended) show clearly that, at high frequencies, the bulk flow velocity in the x-direction, v_x, and the xz-component of the stress tensor, P_{xz}, are zero (except very close to the borders, due to channel end effects) and the other macroscopic quantities $(v_z$ and $P_{zz})$ do not depend on x.

As frequencies decrease, the two-dimensional character of the flow field can not be neglected and a full 2D description becomes mandatory in order to capture the correct magnitude of the macroscopic fields.


δ=0.1
T=0.2
ωt=0
ωt=π/2
ωt=π
ωt=3π/2
v_x
v_z
P_xz
P_zz
x


δ=0.1 T=0.5
v_x
v_z
P_{xz}
P_{zz}
x
ωt=0
ωt=π/2
ωt=π
ωt=3π/2


δ=0.1 T=10.
v_x
v_z
P_xz
P_zz
x
ωt=0
ωt=π/2
ωt=π
ωt=3π/2

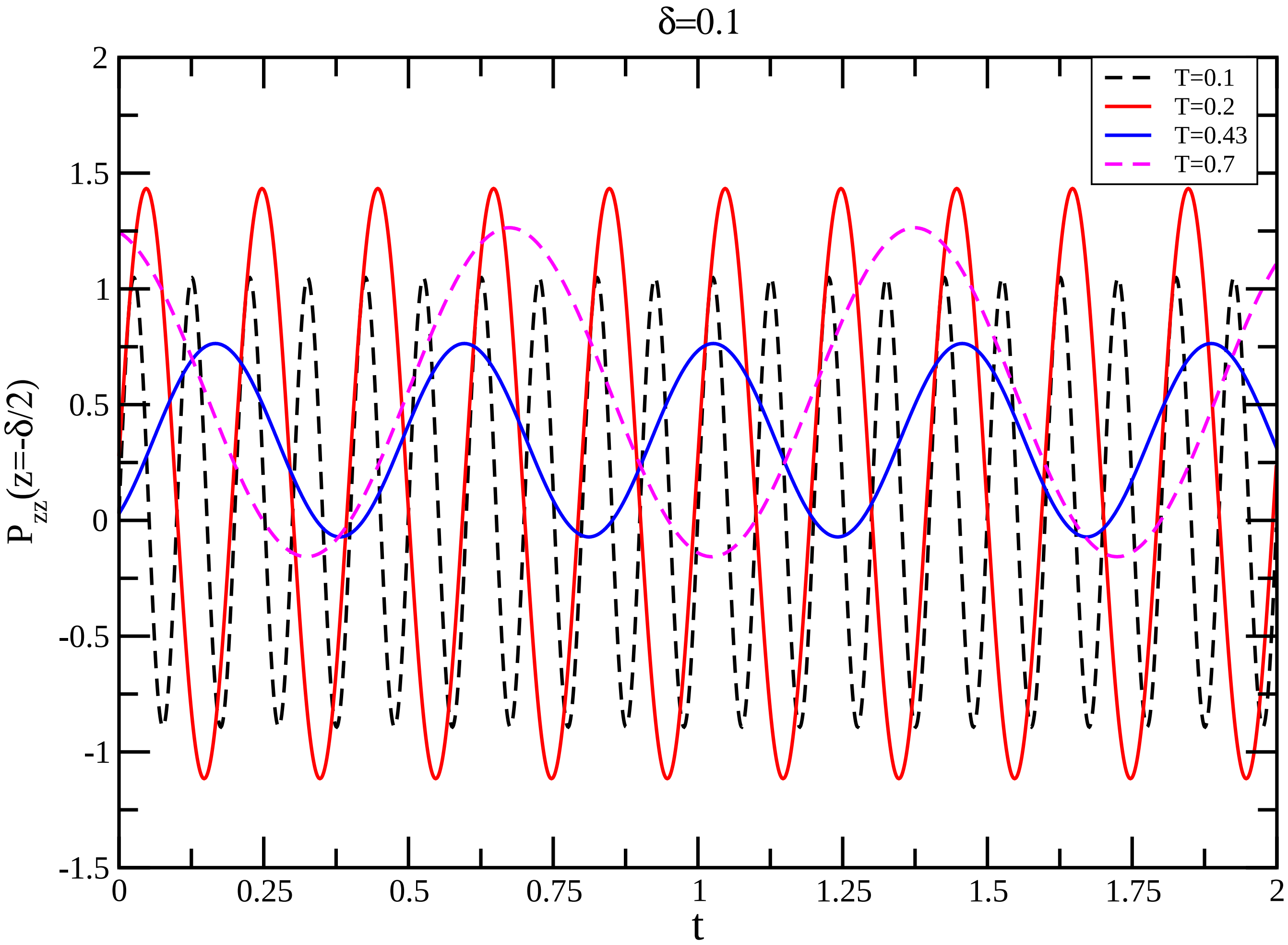
Low frequency MEMS devices are normally operated at very low pressure in order to minimize the damping due to gas flow in the small gaps between the moving parts of these microstructures. This need can be overcome when MEMS devices vibrate at relatively high frequencies, since gas compressibility and inertial forces lead then to another damping mechanism (in addition to the viscous damping that dominates at low frequencies).

In particular, inertial forces will cause a gas resonance in the z-direction when the dimensionless distance between the channel walls (measured in units of the oscillation period of the moving plate)

$$L = \frac{\delta}{v_0\, T} = \frac{d\, \omega'}{2\, \pi\, v_0} \tag{51}$$

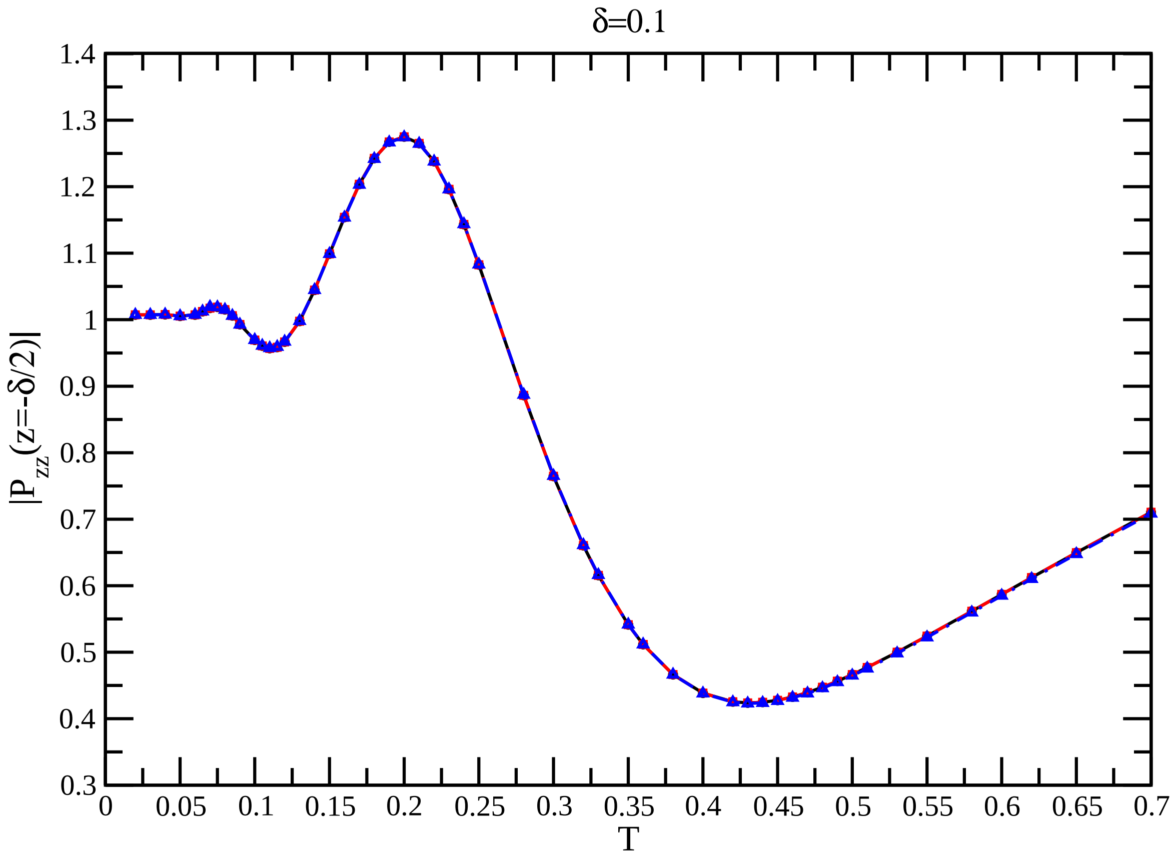
takes a well-defined fixed value (with v_0 being the thermal velocity $\sqrt{2RT_0})$.

Note that the quantity $(2\, \pi\, v_0/\omega')$ is the distance traveled by gaseous molecules during one cycle of oscillation of the moving boundary.



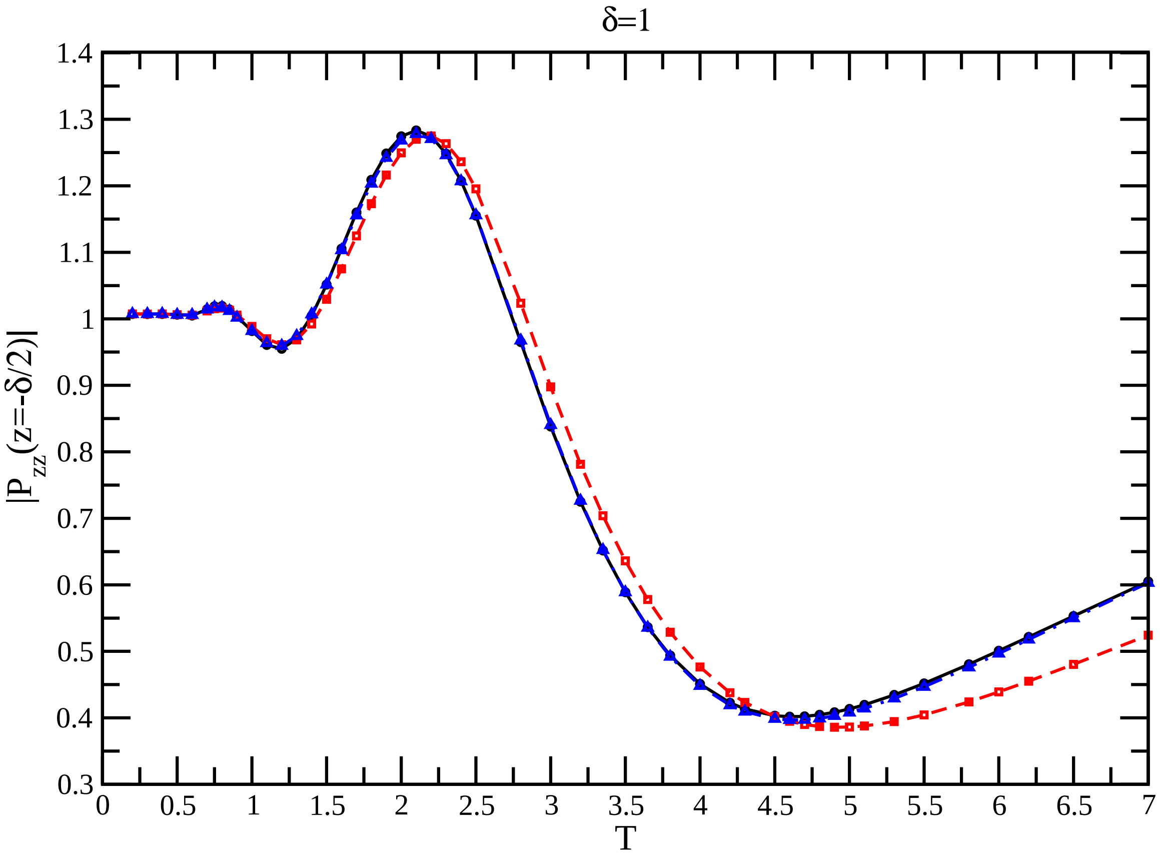
In correspondence with a resonant response of the system, the amplitude of P_{zz} at $z = -\delta/2$ reaches its maximum value (resonance) or its minimum value (antiresonance).

Profiles of the normal stress amplitude at the moving wall



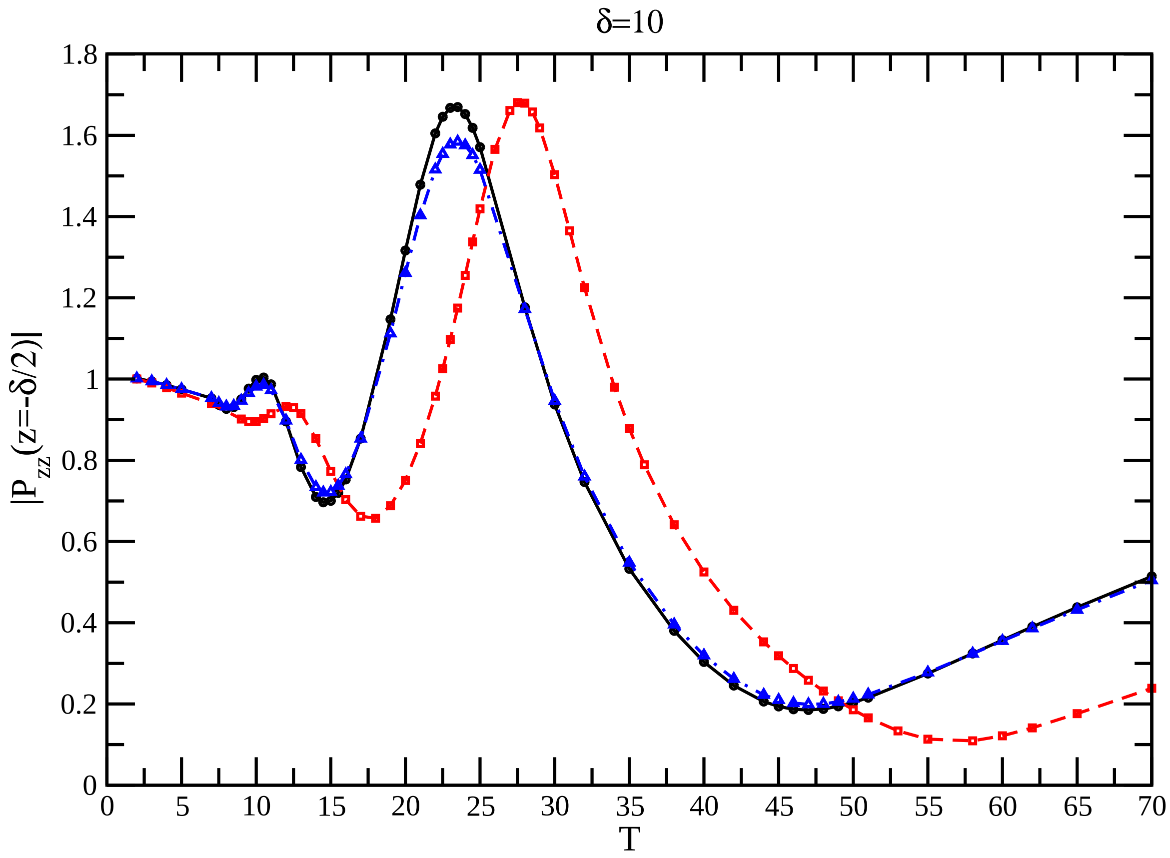
- circles $\longrightarrow$ BGK model
- triangles $\longrightarrow$ Ellipsoidal Statistical (ES) model
- squares $\longrightarrow$ BGK model without thermal terms

The thermal effects do not play any role at high frequency $T \leq 1$.



- circles $\longrightarrow$ BGK model
- triangles $\longrightarrow$ Ellipsoidal Statistical (ES) model
- squares $\longrightarrow$ BGK model without thermal terms

The damping force has a minimum at the first antiresonance frequency.



- circles ⟶ BGK model
- triangles ⟶ Ellipsoidal Statistical (ES) model
- squares ⟶ BGK model without thermal terms

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