Stable determination of an inhomogeneous inclusion in a layered medium

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July 28, 2020

Abstract

In this paper we study the stability issue of the inverse problem of determining an inclusion contained in a stratified medium assuming the conductivity to be isotropic and variable coefficients. We prove a modulus of continuity of logarithmic type which, according to examples showed in [DC-Ro], turns out to be optimal.

1 Introduction

In this paper we consider the inverse problem of determining an inclusion D contained in an electrical conductor Ω. This is a special instance of the inverse conductivity problem proposed by Calderón [Ca]. Namely the inclusion D represents a region of the domain whose conductivity is different from the known conductivity of the background medium. From the viewpoint of applications D may represent an internal part of the body that is going through a process of deteriorating or some part that is changing due to some chemical reaction. One might think to geophysical prospection or medical imaging, for instance.

Keeping in mind the geophysical application, we will consider a stratified medium Ω, where the conductivity is different in each layer. The conductivity is known in each layer and an anomalous region D, whose conductivity is different and unknown, is contained in one layer. The major difficulty, with respect to the previously treated cases, is the presence of a jump interface separating the unknown inclusion from the exterior boundary, where measurements can be collected.

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We consider a layered medium Ω with two layers divided by an interface Σ. We call the two layers Ω_- and Ω_+, where Ω_+ is the inner layer whose boundary is Σ. The conductivity of Ω is known and it is different in each layer. Precisely it is equal to $a_1(x)$ in Ω_- and $a_2(x)$ in Ω_+. Therefore functions a_1 and a_2 are known. Inside the inner layer Ω_+ and anomalous region D, whose conductivity is denoted by $b(x)$ is located. Both D and $b(x)$ are not known as they represent a damage part of the layer whose shape, location and physical characteristics are not available data. Denoting by γ the conductivity of Ω with the presence of the damaged part D, $\gamma(x)$ will be of the form

$$\gamma(x) = a_1(x) + (a_2(x) - a_1(x))\chi_{\Omega_+} + (b(x) - a_2(x))\chi_D,$$

where $a_1(x), a_2(x)$ are known function, whereas $b(x)$ is not known.

Applying a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary of Ω, the induced potential u is the solution of the boundary value problem

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The conormal derivative of the solution on the boundary, $\gamma(x)\nabla u(x)\cdot\nu(x)$,where $\nu(x)$ is the exterior unit normal to $\partial\Omega$ at a point x, is the current density that we measure. By uniqueness of the solution of (1.1) we can define the following operator Λ_D, known as tha Dirichlet–to–Neumann map, as

$$\begin{array}{rcccl} \Lambda_D & : & H^{1/2}(\partial\Omega) & \longrightarrow & H^{-1/2}(\partial\Omega) \\ & & f & \longrightarrow & \gamma(x)\dfrac{\partial u}{\partial \nu}, \end{array}$$

which, from a physical perspective corresponds to performing all possible boundary electrostatic measurements.

The inverse problem we are addressing to, is to recover information on the inclusion D from a knowledge of the Dirichlet–to–Neumann map Λ_D. Uniqueness has been obtained by Isakov [Is88]. Our purpose is to study the stability issue. Precisely we want to determine how the inclusion D depends on the given data Λ_D. These type of problems are usually not well posed in the Hadamard sense, but if some a priori information on the unknown is available, it is possible to provide a modulus of continuity. A first approach in this direction was proposed in [Al-DC] that shows that for piecewise constant conductivity the inclusion D depends on the Dirichlet–to–Neumann map with a modulus of continuity of logarithmic type. This argument has been then generalized to variable coefficient conductivities in [DC]. In [DC-Re] the framework of anisotropic conductivities has been considered. Partial results have been obtained but they are still very limited and it is still an open issue. In [DC-Re2] the analysis of layered media has been performed. The authors consider a piecewise conductivity known in each layer and unknown inside the inclusion.

In this note our purpose is to generalize [DC-Re2] considering variable coefficients conductivities. The argument to obtain stability deals basically with two main issues: quantitative estimates of unique continuation and singular solutions. The first topic can be derived directly from the constant coefficient case [DC-Re2] where the authors make use of a three region type inequality developed in [Fr-Li-Ve-Wa] based on an ad hoc Carleman estimate [DC-Fr-Li-Ve-Wa]. We refer also the interested reader to [DC-Re3] and [Ca-Wa] for recent developments in this direction. The main difficulty is related with the use of singular solutions. In particular it is crucial establishing the asymptotic behavior. More precisely the key part is to compare the fundamental solution of our operator with the fundamental solution of the Laplace operator.

The paper is organized as follows. In the next Section 2 we will introduce our notations and the stability theorem. The proof of it can be found in Section 3 and it is based on some auxiliary propositions proved in Section 4.

2 Main Result

Let us first premise some notations and definitions. For points $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. Moreover, denoted by $\mathrm{dist}(\cdot, \cdot)$ the standard Euclidean distance, we define

$$B_r(x) = \{y \in \mathbb{R}^n \,:\, \mathrm{dist}(x, y) < r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{n-1} \,:\, \mathrm{dist}(x', y') < r\}$$

as the open balls with radius r centered at x and x' respectively. For simplicity, we use B_r, B'_r instead of $B_r(0), B'_r(0')$ respectively.

Definition 2.1. *Let Ω be the bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0, 1]$, we say a portion S of $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r, L > 0$ if for any point $p \in S$, there exists a rigid transformation $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of coordinates under which we have $p = 0$ and*

$$\Omega \cap B_r = \{(x', x_n) \in B_r \,:\, x_n > \varphi(x')\},$$

where $\varphi(\cdot)$ is a $C^{1,\alpha}$ function on B'_r, which satisfies

$$\varphi(0) = |\nabla\varphi(0)| = 0 \qquad \textit{and} \qquad ||\varphi||_{C^{1,\alpha}(B'_r)} \le Lr,$$

where the norm is defined as

$$||\varphi||_{C^{1,\alpha}(B'_r)} := ||\varphi||_{L^\infty(B'_r)} + r||\nabla\varphi||_{L^\infty(B'_r)} + r^{1+\alpha}|\nabla\varphi|_{\alpha,B'_r}$$

$$|\nabla\varphi|_{\alpha,B'_r} := \sup_{\substack{x',y'\in B'_r,\\ x'\neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x'-y'|}.$$

Let us list our main assumptions. Given constants $r_1, M_1, M_2, \delta_1, \delta_2 > 0$ and $0 < \alpha < 1$, we assume the domain $\Omega \subset \mathbb{R}^n$ is bounded

$$|\Omega| \leq M_2 r_1^n \tag{2.1}$$

where $|\cdot|$ denotes the Lebesgue measure.

The interface Σ is C^2 hypersurface with constants r_0, M_0 and $\text{dist}(\Sigma, \partial\Omega) \geq \delta_2$. The inclusion D such that $\text{dist}(D, \Sigma) \geq \delta_1$ and $\Omega\backslash\overline{D}$ is connected. Both ∂D and $\partial\Omega$ are of $C^{1,\alpha}$ class with constants r_1, M_1.

The conductivity $\gamma(x) = a_1(x) + (a_2(x) - a_1(x))\chi_{\Omega_+} + (b(x) - a_2(x))\chi_D$ satisfies the ellipticity condition $\varphi \leq \gamma(x) \leq \varphi^{-1}, \varphi \in (0,1)$ for every $x \in \overline{\Omega}$. Functions a_1, a_2 are of class $C^{0,1}(\overline{\Omega})$ and strictly positive, whereas $b \in C^\alpha(\overline{\Omega})$ strictly positive. Furthermore, there exist positive constants M_{a_1}, M_{a_2}, M_b such that

$$||a_1||_{C^{0,1}(\overline{\Omega})} \leq M_{a_1}, \quad ||a_2||_{C^{0,1}(\overline{\Omega})} \leq M_{a_2}, \quad ||b||_{C^\alpha(\overline{\Omega})} \leq M_b. \tag{2.2}$$

In addition, we assume that there exist a constant $\eta_0 > 0$ such that

$$(a_2(x) - b(x))^2 \geq \eta_0^2 > 0.$$

We refer to $n, r_0, r_1, M_0, M_1, M_2, M_{a_1}, M_{a_2}, M_b, \varphi, \alpha, \delta_1, \delta_2, \eta_0$, as the a priori data. To study the stability, we denote by D_1 and D_2 two inclusions in Ω, which satisfy the above properties and the associated Dirichlet-to-Neumann map as Λ_{D_1} and Λ_{D_2}.

Theorem 2.2. *Let Ω, γ, D_1 and D_2 satisfy the above assumptions. If for some $\varepsilon \in (0,1)$ we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \leq \varepsilon,$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C|\ln t|^{-\eta}, \quad , \forall\, t \in (0,1)$$

and $C > 0$, $0 < \eta \leq 1$ are constants depending on the a priori data only.

3 Proof of the Main Result

The proof of Theorem 2.2 is based on some auxiliary propositions whose proofs are collected in the next Section 4. We denote by $\mathcal{G}$ the connected component of $\Omega_+\backslash(D_1 \cup D_2)$, whose boundary contains Σ. $\Omega_D = \Omega_+\backslash\overline{\mathcal{G}}$, $S^{2r} := \{x \in \mathbb{R}^n : r \leq \text{dist}(x, \Omega) \leq 2r\}$, $S_r := \{x \in \mathcal{C}\Omega : \text{dist}(x, \Omega) \leq r\}$ and $\mathcal{G}^h := \{x \in \mathcal{G} : \text{dist}(x, \Omega_D) \geq h\}$. We recall that the layer Σ separates the domain into two parts Ω_- and Ω_+. We also define $\mathcal{F}^\lambda := \{x \in \Omega_- : \text{dist}(x, \Sigma) \geq \lambda\}$, and $\Sigma_\lambda := \{x \in \Omega_- : \text{dist}(x, \Sigma) = \lambda\}$

We introduce a variation of the Hausdorff distance called the *modified distance*, which simplifies our proof.

Definition 3.1. *The modified distance between D_1 and D_2 is defined as*

$$d_m(D_1, D_2) := \max\left\{ \sup_{x \in \partial\Omega_D \cap \partial D_1} dist(x, D_2), \sup_{x \in \partial\Omega_D \cap \partial D_2} dist(x, D_1) \right\}.$$

With no loss of generality, we can assume that there exists a point $O \in \partial D_1 \cap \partial\Omega_D$ such that the maximum of $d_m = d_m(D_1, D_2) = \text{dist}(O, D_2)$ is attained. We remark here that d_m is not a metric, and in general, it does not dominate the Hausdorff distance. However, under our a priori assumptions on the inclusion, the following lemma holds.

Lemma 3.2. *Under the assumptions of Theorem 2.2, there exists a constant $c_0 \geq 1$ only depending on M_1 and α such that*

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m(D_1, D_2). \tag{3.1}$$

Proof. See [Al-DC, Proposition 3.3] □

At this point we wish to drive the reader attention to the fact that due to the iterative use of the three spheres inequality argument in the proof of our result, it is important to be able to control the radii of such balls in order to remain inside $\mathcal{G}$. This issue was first raised and solved by [Al-Si]. We refer to [Al-DC-Mo-Ro, Lemma 4.2] for a detailed analysis in this direction.

In order to use the information provided by the boundary measurements to evaluate the distance between the two inclusions D_1 and D_2, we apply the following identity firstly introduced by Alessandrini [Al]. Let $u_i \in H^1(\Omega)$, $i = 1, 2$, be solutions to (1.1) with conductivities $\gamma_{D_i} = a_1(x) + (a_2(x) - a_1(x))\chi_{\Omega_+} + (b(x) - a_2(x))\chi_{D_i}$ respectively, we have

$$\int_\Omega (\gamma_{D_1} \nabla u_1 \cdot \nabla u_2) - \int_\Omega (\gamma_{D_2} \nabla u_1 \cdot \nabla u_2) = \int_{\partial\Omega} u_1 [\Lambda_{D_1} - \Lambda_{D_2}] u_2. \tag{3.2}$$

Denoting by Γ_{D_i} the fundamental solution of the operator $\operatorname{div}(\gamma_{D_i}\nabla\cdot)$, for $i=1,2$, we consider (3.2) replacing $u_1(x)$ with $\Gamma_{D_1}(x,y)$ and $u_2(x)$ with $\Gamma_{D_2}(x,z)$, where the singularities y and z are taken in $\mathcal{C}\Omega$, $\mathcal{C}\Omega$ being the complement set of Ω. Defining

$$S_{D_1}(y,z)=\int_{D_1}(b(x)-a_2(x))\nabla\Gamma_{D_1}(x,y)\cdot\nabla\Gamma_{D_2}(x,z)dx$$

$$S_{D_2}(y,z)=\int_{D_2}(b(x)-a_2(x))\nabla\Gamma_{D_1}(x,y)\cdot\nabla\Gamma_{D_2}(x,z)dx$$

$$f(y,z)=S_{D_1}(y,z)-S_{D_2}(y,z),$$

identity (3.2) can be written as

$$f(y,z)=\int_{\partial\Omega}\Gamma_{D_1}(x,y)[\Lambda_{D_1}-\Lambda_{D_2}]\Gamma_{D_2}(x,z),\quad \forall y,z\in\mathcal{C}\Omega. \tag{3.3}$$

The following two propositions provide quantitative estimates on f and S_{D_1}, when moving y and z towards O along $\nu(O)$, where the point O has been defined in Definition 3.1 and it is the point in which the modified distance d_m between D_1 and D_2 is achieved. Their proof are postponed in the next Section 4.

Proposition 3.3. *Given $\varepsilon\in(0,1)$ and a transformation of coordinates defined as $y=h\nu(O)$, if*

$$||\Lambda_{D_1}-\Lambda_{D_2}||_{L(H^{1/2},H^{-1/2})}<\varepsilon, \tag{3.4}$$

then for every h, where $0<h<cr_1, 0<c<1$, and c depending on M_1, we have

$$|f(y,y)|\leq C_0\frac{\varepsilon^{Bh^F}}{h^T}. \tag{3.5}$$

Here $0<T<1$ and $C_0,B,F>0$ are constants that depend only on the a priori data.

Proposition 3.4. *Given a transformation of coordinates $y=h\nu(O)$ defined as above, for every $0<h<r_0/2$*

$$|S_{D_1}(y,y)|\geq C_1h^{2-n}-C_2d_m^{2-2n}+C_3, \tag{3.6}$$

where $r_0:=\frac{r_1}{2}\min\left[\frac{1}{2}(8M_1)^{-1/\alpha},\frac{1}{2}\right]$, and C_1,C_2,C_3 are positive constants depending only on the a priori data.

Now, we have all the ingredients to conclude this section with the proof of Theorem 2.2.

Proof of Theorem 2.2. We start from the origin of the coordinate system, point $O \in \partial D_1 \cap \partial\Omega_D$, for which the maximum in Definition 3.1 is attainted

$$d_m := d_m(D_1, D_2) = \mathrm{dist}(O, D_2).$$

Then, with a transformation of coordinates $y = h\nu(O)$, where $0 < h < h_1, h_1 := \min\{d_m, cr_1, r_0/2\}, 0 < c < 1$, c depending on M_1, applying (2.2) and the inequality $|\nabla_x \Gamma_{D_i}(x,y)| \leq k|x-y|^{1-n}$ (see Theorem 4.1 below), where $k > 0$ is a constant depending only on $\varphi, n, \alpha, M_{a_1}, M_{a_2}, M_b, M_1$, we have

$$\begin{aligned} |S_{D_2}(y,y)| &= \left| \int_{D_2} (b(x) - a_2(x)) \nabla\Gamma_{D_1}(\cdot, y) \nabla\Gamma_{D_2}(\cdot, y) \right| \\ &\leq C_1 \int_{D_2} |x-y|^{2-2n} \leq C_2 \int_{D_2} (d_m - h)^{2-2n} \\ &\leq C_3 (d_m - h)^{2-2n} |D_2| \leq C_4 (d_m - h)^{2-2n}. \end{aligned} \tag{3.7}$$

Here $|D_2|$ is the measure of the inclusion D_2 and C_4 depends on $\varphi, n, \alpha, M_1, M_2, r_1, M_{a_1}, M_{a_2}, M_b$. From (3.5) we have

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \leq |S_{D_1}(y,y) - S_{D_2}(y,y)| = |f(y,y)| \leq C_0 \frac{\varepsilon^{Bh^F}}{h^T}, \tag{3.8}$$

and by Proposition 3.4 and (3.7) we get

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \geq C_5 h^{2-n} - C_6 (d_m - h)^{2-2n}.$$

Thus we obtain

$$C_5 h^{2-n} \leq C_6 (d_m - h)^{2-2n} + C_0 \frac{\varepsilon^{Bh^F}}{h^T},$$

that leads to the inequality

$$C_7 (d_m - h)^{2-2n} \geq C_8 h^{2-n} - \frac{\varepsilon^{Bh^F}}{h^T} = C_9 h^{2-n} (1 - \varepsilon^{Bh^F} h^K),$$

where $0 < K = n - 2 - T$ and $C_9 = \min\{C_8, 1\}$. Now let $h = h(\varepsilon) = \min\left\{ |\ln\varepsilon|^{-\frac{1}{2F}}, d_m \right\}$, for $0 < \varepsilon \leq \varepsilon_1$, where $\varepsilon_1 \in (0,1)$ is chosen so that $\exp(-B|\ln\varepsilon_1|^{1/2}) = 1/2$. If $d_m \leq |\ln\varepsilon|^{-\frac{1}{2F}}$, the theorem follows by Lemma 3.2. Indeed setting $\eta = \frac{1}{2F} > 0$, we have

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m \leq c_0 |\ln\varepsilon|^{-\eta} = \omega(\varepsilon) \tag{3.9}$$

If $d_m \geq |\ln\varepsilon|^{-\frac{1}{2F}}$, then

$$(d_m - h)^{2-2n} \geq C_{10} h^{2-n} \quad \Longrightarrow \quad d_m \leq C_{11} |\ln\varepsilon|^{-\frac{n-2}{4F(n-1)}},$$

where C_{11} depends only on the a priori data. Therefore we conclude the proof by setting $\eta = \frac{n-2}{4F(n-1)}$

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 d_m \leq C_{12} |\ln\varepsilon|^{-\eta} = \omega(\varepsilon). \tag{3.10}$$

When $\varepsilon_1 \leq \varepsilon$, then $d_m \leq \mathrm{diam}(\Omega)$ and, in particular when $\varepsilon_1 \leq \varepsilon < 1$

$$d_m \leq \mathrm{diam}(\Omega) \frac{|\ln\varepsilon|^{-\frac{1}{2F}}}{|\ln\varepsilon_1|^{-\frac{1}{2F}}},$$

and the thesis follows using Lemma 3.2. □

4 Proof of the Auxiliary Propositions

In this section we show the proofs of the propositions used to prove our stability theorem. We start with Proposition 3.3 following the three spheres inequality argument proposed in [Al-DC, Proposition 3.5]. In this physical context, however, a crucial point is when we have to cross the interface Σ. This is possible by applying an ad hoc three region inequality proved in [Ca-Wa].

Proof of Proposition 3.3. Let us consider $f(y, \cdot)$ with a fixed $y \in S^{2r}$ then

$$\mathrm{div}\,(\gamma(\cdot)\nabla f(y, \cdot)) = 0, \text{ in } \mathcal{C}\overline{\Omega}_D. \tag{4.1}$$

For $x \in S^{2r}$, by (3.3) and (3.4), we have

$$|f(y, x)| \leq C(r, M_1, M_2) ||\Lambda_{D_1} - \Lambda_{D_2}|| = C\varepsilon. \tag{4.2}$$

Also by [Al-DC, Proposition 3.4], we have a uniform bound on f

$$|f(y, x)| \leq ch^{2-2n}, \qquad \text{in } \mathcal{G}^h. \tag{4.3}$$

For any $0 < \overline{r} < r$ and for every $\overline{\omega} \in \mathcal{F}^\lambda$, we can have a smallness bound on f on any arbitrarily small ball $B_{\overline{r}/2}(\overline{\omega}) \subset \mathcal{F}^\lambda$ by an iteratively application of the three spheres inequalities on a simple arc $\gamma \in \overline{\Omega}_- \cup \overline{S}_r \cup \overline{S}^{2r}$ which connects $\overline{\omega}$ and x. By (4.21) of [Al-DC], we can reach $\overline{\omega}$ from x with a finite number s of balls. Thus we obtain

$$||f(y, \cdot)||_{L^\infty(B_{\overline{r}/2}(\overline{\omega}))} \leq C(h^{1-n})^{1-\tau^s} \varepsilon^{\tau^s}, \tag{4.4}$$

where C depends on the a priori data.

At this stage to cross the interface Σ, by [Ca-Wa, Theorem 1.1] combined with (4.4) and standard bounds we get

$$||f(y,\cdot)||_{L^\infty(B_{r_0/2}(x_0))} \leq C(h^{1-n})^{\mathcal{A}}\varepsilon^{\mathcal{B}}, \tag{4.5}$$

where $x_0 = (0, -H)$ is a point in Ω_+ with $H > 0$ depending on the a priori data, and $\mathcal{A}$ and $\mathcal{B}$ are positive constants depending on the a priori data.

The rest of the propagation is similar as (4.4). If we choose any $0 < \overline{r_0} < r_0$ and any $\overline{\omega}_0 \in \Omega_+$, by connecting x_0 and $\overline{\omega}_0$ with a simple arc, we obtain

$$||f(y,\cdot)||_{L^\infty(B_{\overline{r}_0/2}(\overline{\omega}_0))} \leq C(h^{1-n})^{\overline{\mathcal{A}}}\varepsilon^{\overline{\mathcal{B}}}. \tag{4.6}$$

The rest of the proof follows by (4.22)–(4.25) of [Al-DC]: we define a truncated cone $C(O, \nu(O), \theta, r)$, in which $O \in D_1$ is the point where the maximum of Definition 3.1 is attained. Then we consider $f(y, w)$ as a function of y to obtain similar results. The last step is to choose $y = w = h\nu(O)$, where $\nu(O)$ is the exterior unit normal to $\partial\Omega_D$ in O, we can obtain

$$|f(y,y)| \leq Ch^T(\varepsilon^{\tilde{B}A^{k(h)-1}})^{\gamma A^{k(h)-1}}. \tag{4.7}$$

We observe that for $0 < h < cr$, where $0 < c < 1$ depends on M_1, $k(h) \leq c|\ln A| = -c\ln h$, we can rewrite

$$A^{k(h)} = \exp\{-c\ln h \ln A\} = h^{-c\ln A} = h^{c|\ln A|} = h^Q,$$

$$(A^{k(h)})^2 = (h^Q)^2 = h^F$$

with $F = 2Q = 2c|\ln A|$. Therefore

$$\begin{aligned} |f(y,y)| &\leq Ch^{-T}\varepsilon^{B(A^{k(h)})^2} \\ &\leq \exp\{-T\ln h\}\exp\{B(A^{k(h)})^2\ln\varepsilon\} \\ &\leq \exp\{-T\ln h + Bh^F\ln\varepsilon\} \\ &= \frac{\varepsilon^{Bh^F}}{h^T}, \end{aligned}$$

where $B = \frac{\gamma\tilde{B}}{A^2}$ □

The proof of Proposition 3.4 can be obtained along the lines of [DC, Proposition 3.5] through minor adaptations. Let us just point out that a crucial step of the argument is an asymptotic analysis of Γ_D. In particular, denoting by χ_+ the characteristic function of the half space $\{x_n > 0\}$, $b_0 = b(0)$, $a_0 = a_2(0)$ and Γ_+ the fundamental solution of the operator $\mathrm{div}((a_0 + (b_0 - a_0)\chi_+)\nabla\cdot)$, the following theorem holds (see [DC, Proposition 3.3]).

Theorem 4.1. *Let $D \in \mathbb{R}^n$ be an open set with $C^{1,\alpha}$ boundary with constants M_1, r. For any $x, y \in \mathbb{R}^n$, there exists a positive constant C_1, depending on the a priori data such that*

$$|\nabla \Gamma_D(x,y)| \leq C_1 |x-y|^{1-n}.$$

Also for $x \in D \cap B_\rho(O)$ and $y = h\nu(O)$, with $0 < \rho < r_0$ and $0 < h < r_0$ where $r_0 = \frac{r}{2}\min\{\frac{1}{2}(8M_1)^{-1/2}, \frac{1}{2}\}$ we have

$$|\Gamma_D(x,y) - \Gamma_+(x,y)| \leq \frac{C_2}{r^\alpha}|x-y|^{\alpha-n+2}$$

$$|\nabla \Gamma_D(x,y) - \nabla \Gamma_+(x,y)| \leq \frac{C_3}{r^{\alpha^2}}|x-y|^{\alpha^2-n+1},$$

where $C_i > 0$, $i = 2, 3$, depend on the a priori data and $0 < \alpha < 1$.

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