Three Sphere Inequality for Second Order Elliptic Equations with Coefficients with Jump Discontinuity

Michele Di Cristo and Yong Ren*

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Abstract

This is a short note to complete the paper appeared in J. Differential Equations 261 (2016), no. 10, pp. 5306–5323, where a rough version of the classical well known Hadamard three–circle theorem for solution of an elliptic PDE in divergence form has been proved. Precisely, instead of circles, the authors obtain a similar inequality in a more complicated geometry. In this paper we clean the geometry and obtain a generalized version of the three-circle inequality for elliptic equation with coefficients with discontinuity of jump type.

1 Introduction

In this note we consider a generalization of the Hadamard three-circles theorem to solution of a divergence form elliptic equation in $\mathbb{R}^n$ with discontinuous coefficients. Motivated by the study of the inverse problem of determining an inclusion D in an electrical conductor Ω, the physical situation we aim to analyze is a layered medium, where each layer has a known conductivity, with a region D, whose conductivity is different from the surrounding material, located inside. Therefore, denoting by $A(x)$ the conductivity, A turns out to be a piecewise constant function.

We are interested in obtaining a three spheres inequality of the form

$$\|u\|_{L^\infty(B_{r_2})} \leq C\|u\|^{\tau}_{L^\infty(B_{r_1})}\|u\|^{1-\tau}_{L^\infty(B_{r_3})}, \tag{1.1}$$

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*Politecnico di Milano, Italy (`michele.dicristo@polimi.it, yong.ren@polimi.it`).

for solution u of elliptic equation

$$\operatorname{div}(A(x)\nabla u) = 0, \qquad \text{in } \Omega,$$

where B_{r_i}, $i = 1, 2, 3$, is the ball of radius r_i centered at any point $x \in \Omega \setminus D$, $0 < r_1 < r_2 < r_3$ and $\tau \in (0, 1)$.

This is a classical tool in PDEs that provides an estimate of the norm of the solution in a middle ball in term of its norm in a smaller ball and in a larger ball. This property, established by Hadamard for harmonic functions, has been obtained by Landis [La] for L^∞–norms and Agmon for L^2–norms for solutions of general elliptic PDEs with smooth coefficients. Later refinements can be found in [Ko-Me, Br, Ku]. Recently the case with coefficients with jumps has been considered. In particular in [Fr-Li-Ve-Wa] a weaker version of (1.1) is obtained. Namely the authors prove a similar inequality for L^2 norms in a more complicated geometry instead of balls. A crucial tool to get this inequality is a suitable Carlemann estimate obtained in [DC-Fr-Li-Ve-Wa], where the second order elliptic operator considered has discontinuous coefficients with discontinuities that occur as jump at the interface. Let us mention here several closely related papers as [LR-Ro, LR-Ro2, LR-Le].

In this paper we proceed along this line refining the geometry of the inequality obtained in [Fr-Li-Ve-Wa] and getting a three sphere inequality.

These tools are important in application to inverse problems as they allow to evaluate quantitatively how some quantity propagates inside a domain. Specific applications can be found in [Fr-Li-Ve-Wa], where size estimates for unknown inclusions are proved, and in [DC-Re] where stability estimates for the inverse inclusion problem is studied.

In the next Section 2 we will state three sphere type theorem specifying the hypothesis needed. The proof is provided in Section 3 where the Carlemann estimate and the three region inequality used are recalled.

2 Assumptions and Main Result

In this Section we state our main result. We start by fixing some notations and listing the hypothesis we need. We denote by Ω a bounded open set in $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary $\partial\Omega$ with constants s_0, L_0, where $0 < \alpha \leq 1$, such that $|\Omega| \leq C r_0^n$, for some given $r_0 > 0$ with C a positive constant. Assume that Σ is a $C^{1,1}$ hypersurface with constants s_1, L_1 that divides Ω into two open sets Ω_+ and Ω_- such that

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-.$$

Denoting by $H_{\pm}^{(\Omega)} = \chi_{\Omega_{\pm}}$, we consider the conductivity equation

$$\mathrm{div}(A\nabla u) = 0, \qquad \text{in } \Omega, \tag{2.1}$$

where $A = H_{+}^{(\Omega)}A_{+} + H_{-}^{(\Omega)}A_{-}$ with

$$A_{\pm}(x) = \{a_{ij}^{\pm}(x)\}_{i,j=1}^{n}, \quad x \in \mathbb{R}^n$$

a Lipschitz symmetric matrix-valued function satisfying for given constants $\lambda \in (0,1]$, $\Lambda > 0$

$$\lambda|z|^2 \leq A_{\pm}(x)z \cdot z \leq \lambda^{-1}|z|^2, \quad \forall\, x \in \mathbb{R}^n, \forall\, z \in \mathbb{R}^n \tag{2.2}$$

and

$$|A_{\pm}(x') - A_{\pm}(x)| \leq \frac{\Lambda}{r_0}|x' - x|. \tag{2.3}$$

We can now state our main theorem.

Theorem 2.1. *Let u be a solution to* (2.1) *and $A_{\pm}(x)$ satisfy* (2.2) *and* (2.3)*. Then there exist $C > 0$ depending on λ, Λ, n such that*

$$\|u\|_{L^\infty(B_{l_1 r}(z))} \leq C\|u\|_{L^\infty(B_r(z))}^{\tau}\|u\|_{L^\infty(B_{l_2 r}(z))}^{1-\tau}, \tag{2.4}$$

for $z \in \Omega \setminus D$, where $0 < \tau < 1$ depends on the a priori data and $1 < l_1 < l_2$ such that $B_{l_2 r}(z) \subset \Omega$, for some $r < 1$.

Remark 2.2. *This result remain valid if we add lower order terms of the form $\sum_{\pm} H_{\pm}(W\nabla u + Vu)$, where W, V are bounded function, to* (2.1)*. Its proof, indeed, makes use of an estimate that holds true for more general operators (see [Fr-Li-Ve-Wa, Remark 2.2]).*

3 Proof of Theorem 2.1

In this section we provide the proof of Theorem 2.1. Without loss of generality, we can assume that the interface Σ is planar. Indeed, since Σ is $C^{1,1}$, for any $P \in \Sigma$ there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega_{\pm} \cap B_{r_0}(0) = \{(x,y) \in B_{r_0}(0) \subset \mathbb{R}^n \,:\, y \gtrless \psi(x)\},$$

where ψ is a $C^{1,1}$ function on $B'_{r_0}(0) \subset \mathbb{R}^{n-1}$ satisfying $\psi(0) = 0$ and $\|\psi\|_{C^{1,1}(B'_{r_0}(0))} \leq K_0$. Using the coordinate transform $(x', y') = T(x,y) = (x, y - \psi(x))$ for $x \in B'_{r_0}$, we reduce our analysis to the planar interface.

Therefore we will prove Theorem 2.1 assuming Σ to be planar. We denote by $H_\pm = \chi_{\mathbb{R}^n_\pm}$, where $\mathbb{R}^n_\pm = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, y \gtrless 0\}$. Let $u_\pm \in C^\infty(\mathbb{R}^n)$ and set

$$u = H_+u_+ + H_-u_- = \sum_\pm H_\pm u_\pm,$$

we define

$$\mathcal{L}u := \sum_\pm H_\pm \mathrm{div}(A(x,y)\nabla u_\pm). \tag{3.1}$$

To prove Theorem 2.1 we will make use of the following three–region inequality.

Theorem 3.1. *Let u be a solution of* (3.1)*. There exist C and R depending on the a priori data such that if $0 < R_1,\ R_2 < R$, then*

$$\int_{U_2} |u|^2 dx \leq C \left(\int_{U_1} |u|^2 dxdy \right)^{\frac{R_2}{2R_1+3R_2}} \left(\int_{U_3} |u|^2 dxdy \right)^{\frac{2R_1+2R_2}{2R_1+3R_2}}, \tag{3.2}$$

where

$$\begin{aligned} U_1 &= \{-4R_2 \leq z(x,y),\ \ \frac{R_1}{8a} < y < \frac{R_1}{a}\}, \\ U_2 &= \{-R_2 \leq z(x,y) \leq \frac{R_1}{2a},\ \ y < \frac{R_1}{8a}\}, \\ U_3 &= \{-4R_2 \leq z(x,y),\ \ y < \frac{R_1}{a}\}, \end{aligned} \tag{3.3}$$

$a = \alpha_+/\delta$ and

$$z(x,y) = \frac{\alpha_-}{\delta}y + \frac{\beta}{2\delta^2}y^2 - \frac{1}{2\delta}|x|^2.$$

For the proof of this, we refer to [Fr-Li-Ve-Wa, Theorem 3.1]. Let us only mention that it is based on a proper use of a Carlemann estimate obtained in [DC-Fr-Li-Ve-Wa].

Let us now denote some parameters to describe the geometric properties of the regions. We use l_1, l_2, l_3 to represent the longest "length" for regions U_1, U_2, U_3 along x-axis. We use d_1, d_2, d_3 to represent the longest "depth" for regions U_1, U_2, U_3 along y-axis. With some calculations, we obtain

$$\begin{aligned}
l_1 = l_3 &= 2\sqrt{\frac{\beta}{\delta}\left(\frac{R_1}{a}\right)^2 + 2\alpha_-\frac{R_1}{a} + 8\delta R_2} \\
l_2 &= 2\sqrt{\frac{\beta}{\delta}\left(\frac{R_1}{8a}\right)^2 + 2\alpha_-\frac{R_1}{8a} + 2\delta R_2} \\
d_1 &= \frac{7R_1}{8a} \\
d_2 &= \frac{R_1}{8a} + \frac{\delta}{\beta}\left(\alpha_- - \sqrt{\alpha_-^2 - 2\beta R_2}\right) \\
d_3 &= \frac{R_1}{a} + \frac{\delta}{\beta}\left(\alpha_- - \sqrt{\alpha_-^2 - 8\beta R_2}\right)
\end{aligned} \tag{3.4}$$

Proof of Theorem 2.1. For any point $O \in \Omega \setminus D$, we build a coordinator system x-O-y. First, we want to have $U_1 \subset B_{r_1}$. Then, we will use a finite union of U_2 to cover B_{r_2}, that is, there exists $M < \infty$, such that $B_{r_2} \subset \cup_{j=1}^{M} U_{2j}$. Finally, we want $\cup_{j=1}^{M} U_{3j} \subset B_{r_3}$. All these can be done by choosing the proper R_1, R_2, a, i.e., the proper geometric structures for these regions.

(i) $U_1 \subset B_{r_1}$. We want the longest distance between O and any point in U_1 less then the radius of the B_{r_1}. In this case, it is easy to calculate $(\frac{l_1}{2})^2 + (\frac{R_1}{a})^2 \leq r_1^2$, which gives

$$\left(\frac{\beta}{\delta} + 1\right)\left(\frac{R_1}{a}\right)^2 + 2\alpha_-\frac{R_1}{a} + 8\delta R_2 \leq r_1^2 \tag{3.5}$$

(ii) $B_{r_2} \subset \cup_{j=1}^{M} U_{2j}$. Since the Lebesgue measure of the whole domain $|\Omega|$ is finite. We can always cover B_{r_2} by duplicating a finite amount of U_{2j}, $j = 1, \ldots, M$, along both x-axis and y-axis. In fact, we need at least $\frac{2r_2}{l_2}$ amounts of U_{2j} along x-axis; and at least $\frac{2r_2}{d_2}$ amounts of U_{2j} along the y-axis to cover the whole B_{r_2}. In this case, a wise choice of M should be

$$M = \left\lceil \frac{2r_2}{l_2} \right\rceil \times \left\lceil \frac{2r_2}{d_2} \right\rceil \tag{3.6}$$

where $\lceil \cdot \rceil$ is the ceiling function, which maps any integer to the least integer that is greater or equal to itself.

(iii) $\cup_{j=1}^{M} U_{3j} \subset B_{r_3}$. In the previous step, we use the union of M regions. This will magnify the total "length" and "depth" of the union $\cup_{j=1}^{M} U_{3j}$. We want the longest distance between O and any point in $\cup_{j=1}^{M} U_{3j}$ less than the radius of B_{r_3}. In this case, it is easy to calculate the total "length" of the union $\cup_{j=1}^{M} U_{3j}$ is $l_3 \left\lceil \frac{2r_2}{l_2} \right\rceil$; and the total "depth" of the union $\cup_{j=1}^{M} U_{3j}$ is $d_3 \left\lceil \frac{2r_2}{d_2} \right\rceil$. Thus, the longest distance should be less than r_3, which is

$$\left(l_3 \left\lceil \frac{2r_2}{l_2} \right\rceil\right)^2 + \left(d_3 \left\lceil \frac{2r_2}{d_2} \right\rceil\right)^2 \leq r_3^2 \tag{3.7}$$

Subject to regularities (3.5), (3.7), as well as the geometric relationships; we could apply the three-region inequalities and the standard bound for L^∞ norm (see [Gi-Tr, Chapter 8])

$$\begin{aligned} ||u||_{L^\infty(B_{r_2})} &\leq C||u||_{L^2(B_{r_2})} \leq C||u||_{L^2(\cup_{j=1}^{M} U_{2j})} \\ &\leq CM||u||_{L^2(U_2)} \leq CM||u||^{\gamma}_{L^2(U_1)}||u||^{1-\gamma}_{L^2(U_3)} \\ &\leq C||u||^{\gamma}_{L^2(U_1)}||u||^{1-\gamma}_{L^2(\cup_{j=1}^{M} U_{3j}))} \\ &\leq C||u||^{\gamma}_{L^2(B_{r_1})}||u||^{1-\gamma}_{L^2(B_{r_3})} \\ &\leq C||u||^{\tau}_{L^\infty(B_{r_1})}||u||^{1-\tau}_{L^\infty(B_{r_3})}, \end{aligned} \tag{3.8}$$

where $\|u\|_{L^2(B_r)} = r^n \int_{B_r} |u|^2$ and C depends on λ, Λ

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