Stable Determination of an Inclusion in a Layered Medium with Special Anisotropy

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Abstract In this note we review some recent results concerning the inverse inclusion problem. In particular we analyze the stability issue for defect contained in layered medium where the conductivity is different in each layer. We consider conductivities with special anisotropy. The modulus of continuity obtained is of logarithmic type, which as shown in [DC-Ro] turns out to be optimal.

1 Introduction

In this note we review some recent results related to the inverse problem of determining an inclusion in a conductor body. This is a special instance of the well–known Calderon's inverse conductivity problem [Ca] and it has been studied by Isakov [Is], who shows that the defect can be uniquely recovered through a knowledge of all possible boundary measurements. In this paper the author shows that the defect can be uniquely recovered through a knowledge of all possible electrostatic boundary measurements, making use of the Runge Approximation Theorem and solutions of the governing equation with Green's function type singularities. In 2005 Alessandrini and Di Cristo [Al-DC] have studied the stability issue, that is the continuous dependance of the inclusion from the given data. The approach proposed by the authors is to convert Isakov's idea in a quantitative form. Under mild a priori assumptions on the regularity and the topology of the inclusion, they show that the modulus of continuity of the stability issue is of logarithmic type.

Their result is proved for piecewise constant conductivities and for variable coefficients conductivities [DC]. The argument proposed turns out to be extremely flexible and it has been extended to other physical situations governed by different differential equations. Logarithmic stability estimates hold true for the inverse problem of locating a scattered object by a knowledge of the near field data [DC2], or

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an inclusion in an elastic body by assuming the displacement and the traction on the boundary [Al-DC-Mo-Ro]. These papers are based on an accurate use of the fundamental solution of the differential operator involved and a precise and quantitative evaluation of unique continuation.

These arguments work well in different frameworks with isotropic conductivities in homogeneous conductors but they become more delicate when the physical phenomena take place in a layered medium with anisotropic conductivities. The key items, that cause the main difficulties, are the presence of an unknown boundary (the layer), when we apply the unique continuation technique, and the matrixes, that model the anisotropic conductivities that create big difficulties in estimating the fundamental solution. In several recent results [DC-Re, DC-Re2, DC-Re3, DC-Fr-Li-Ve-Wa, Fr-Li-Ve-Wa] these problems have been considered and some preliminary results in this direction are now available. In this paper we go through these results and summarize the situation showing the state of the art.

The paper is organized as follow. In the next Section 2 we define our notation and state the main theorem. Its proof is presented in Section 3 using some auxiliary results that are proved in Section 4.

2 Notations and Main Result

To begin with, let us premise some notations and definitions, we will use throughout the paper. Let Ω be a bounded open set in $\mathbb{R}^n$ and Σ a layer contained in it. The layer Σ will be a closed hyper-surface that separates Ω in to the union of three parts

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-,$$

where $\Omega_\pm$ are open subsets such that $\partial\Omega_- = \partial\Omega \cup \Sigma$ and $\partial\Omega_+ = \Sigma$. We also denote by D a subset of Ω such that $D \subset \Omega_+ \subset \Omega$. We consider $\gamma(x)$ the conductivity if Ω of the form

$$\gamma_D(x) = c_1 A(x) + (c_2 - c_1)A(x)\chi_{\Omega_+} + (k - c_2)\chi_D,$$

where $A(x)$ is a known Lipischitz matrix valued function satisfying $\|A\|_{C^{0,1}(\Omega)} \leq \overline{A}$ and ellipticity condition with constant $\sigma > 0$, that is

$$\sigma^{-1}|\xi|^2 \leq A(x)\xi\cdot\xi \leq \sigma|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n,$$

c_1 and c_2 are given constants and k is an unknown constant.

For points $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. Moreover, denoted by $\mathrm{dist}(\cdot,\cdot)$ the standard Euclidean distance, we define

$$B_r(x) = \{y \in \mathbb{R}^n | \mathrm{dist}(x,y) < r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{n-1} | \mathrm{dist}(x',y') < r\}$$

as the open balls with radius r centered at x and x' respectively. We write $Q_r(x) = B'_r(x') \times (x_n - r, x_n + r)$ for the cylinder in $\mathbb{R}^n$. For simplicity, we use B_r, B'_r, Q_r instead of $B_r(0), B'_r(0')$ and $Q_r(0)$ respectively. We shall also denote half domain, as well as its associated ball and cylinder

$$\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \quad B^+_r = B_r \cap \mathbb{R}^n_+; \quad Q^+_r = Q_r \cap \mathbb{R}^n_+.$$

Definition 1. Let Ω be the bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0,1]$, we say a portion S of $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r, L > 0$ if for any point $p \in S$, there exists a rigid transformation $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of coordinates under which we have $p = 0$ and

$$\Omega \cap B_r = \{(x', x_n) \in B_r | x_n > \varphi(x')\},$$

where $\varphi(\cdot)$ is a $C^{1,\alpha}$ function on B'_r, which satisfies

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$||\varphi||_{C^{1,\alpha}(B'_r)} \leq Lr,$$

where the norm is defined as

$$||\varphi||_{C^{1,\alpha}(B'_r)} := ||\varphi||_{L^\infty(B'_r)} + r||\nabla\varphi||_{L^\infty(B'_r)} + r^{1+\alpha}|\nabla\varphi|_{\alpha,B'_r}$$

$$|\nabla\varphi|_{\alpha,B'_r} := \sup_{\substack{x',y' \in B'_r, \\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x' - y'|^\alpha}.$$

For $f \in H^{1/2}(\partial\Omega)$, let u be the solution of the problem

$$\begin{cases} \mathrm{div}(\gamma_D(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The inverse problem we addressed to is determine the anomalous region D when the Dirichlet-to-Neumann map Λ_D

$$\begin{array}{rcl} \Lambda_D : H^{1/2}(\partial\Omega) & \longrightarrow & H^{-1/2}(\partial\Omega) \\ f & \longrightarrow & \frac{\partial u}{\partial \nu}_{|\partial\Omega}, \end{array}$$

is given for any $f \in H^{1/2}(\partial\Omega)$. Here, ν denotes the outer unit normal to $\partial\Omega$, and $\frac{\partial u}{\partial \nu}_{|\partial\Omega}$ corresponds to the current density measured on $\partial\Omega$. Thus, the Dirichlet–to–Neumann map represents the knowledge of infinitely many boundary measurements.

Given constants $r_1, M_1, M_2, \delta_1, \delta_2 > 0$ and $0 < \alpha < 1$, we assume the domain $\Omega \subset \mathbb{R}^n$ is bounded

$$|\Omega| \leq M_2 r_1^n,$$

where $|\cdot|$ denotes the Lebesgue measure.

The interface Σ is C^2 and assumed to stay away from the boundary of the domain, as $\mathrm{dist}(\Sigma, \partial\Omega) \geq \delta_2$, and the inclusion D is assumed to stay away from Σ, as $\mathrm{dist}(D, \Sigma) \geq \delta_1$, and also $\Omega \backslash D$ is connected. Both ∂D and $\partial\Omega$ are of $C^{1,\alpha}$ class with constants r_1, M_1.

We refer to $n, r_1, M_1, M_2, \alpha, \delta_1, \delta_2$ as the **a priori data**. To study the stability, we denote by D_1 and D_2 two possible inclusions in Ω, which satisfy the above properties. The associated Dirichlet-to-Neumann maps are Λ_{D_1} and Λ_{D_2}. We also denote by $d_{\mathscr{H}}$ the Housdorff distance between closed sets.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and we have two known constants c_1, c_2 and one unknown constant k, which are given. Let D_1, D_2 be two inclusions in Ω as above. If for any $\varepsilon > 0$ we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathscr{L}(H^{1/2}, H^{-1/2})} \leq \varepsilon,$$

then

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C|\log t|^{-\eta}, \quad \forall t \in (0,1)$$

and $C > 0$, $0 < \eta \leq 1$ are constants depending on the a priori data only.

3 Proof of the Main Result

The proof of Theorem 1 is based on some auxiliary propositions whose proofs are collected in the next Section 4. We denote by $\mathscr{G}$ the connected component of $\Omega \backslash (D_1 \cup D_2)$, whose boundary contains $\partial\Omega$. $\Omega_D = \Omega \backslash \overline{\mathscr{G}}$, $S_{2r} := \{x \in \mathbb{R}^n | r \leq dist(x, \Omega) \leq 2r\}$, $S_r := \{x \in \mathscr{C}\Omega | dist(x, \Omega) \leq r\}$ and $\mathscr{G}^h := \{x \in \mathscr{G} | dist(x, \Omega_D) \geq h\}$, where $\mathscr{C}\Omega$ stands for the complement set of Ω. We recall that the layer Σ separates the domain into two parts known as Ω_- and Ω_+. We also define $\mathscr{F}^\lambda := \{x \in \Omega_- | dist(x, \Sigma) \geq \lambda\}$, and $\Sigma_\lambda := \{x \in \Omega_- | dist(x, \Sigma) = \lambda\}$

We introduce a variation of the Hausdorff distance called the *modified distance*, which simplifies our proof.

Definition 2. The modified distance between D_1 and D_2 is defined as

$$d_m(D_1, D_2) := \max \left\{ \sup_{x \in \partial\Omega_D \cap \partial D_1} dist(x, \partial D_2), \sup_{x \in \partial\Omega_D \cap \partial D_2} dist(x, \partial D_1) \right\}.$$

With no loss of generality, we can assume that there exists a point $O \in \partial D_1 \cap \partial\Omega_D$ such that the maximum of $d_m = d_m(D_1, D_2) = \mathrm{dist}(O, D_2)$ is attainted. We remark here that d_m is not a metric, and in general, it does not dominate the Hausdorff distance. However, under our *a priori* assumptions on the inclusion, the following lemma holds.

Lemma 1. *Under the assumptions of Theorem 1, there exists a constant $c_0 \geq 1$ only depending on M_1 and α such that*

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \leq c_0 d_m(D_1, D_2). \tag{2}$$

Proof. See [Al-DC, Proposition 3.3]

Another obstacle comes from the fact that the propagation of smallness arguments are based on an iterated application of the three spheres inequality for solutions of the equation over chains of balls contained in $\mathscr{G}$. Therefore, it is crucial to control from below the radii of these balls. In the following Lemma 2 we treat the case of points of $\partial\Omega_D$ that are not reachable by such chains of balls. This problem was originally considered by [Al-Si] in the context of cracks detection in electrical conductors.

Let us premise some notations. Given $O = (0, \ldots, 0)$ the origin, v a unit vector, $H > 0$ and $\vartheta \in \left(0, \frac{\pi}{2}\right)$, we denote

$$C(O, v, \vartheta, H) = \{x \in \mathbb{R}^n : \ |x - (x \cdot v)v| \leq \sin\vartheta |x|, \ 0 \leq x \cdot v \leq H\}$$

the closed truncated cone with vertex at O, axis along the direction v, height H and aperture 2ϑ. Given R, d, $0 < R < d$ and $Q = -de_n$, where $e_n = (0, \ldots, 0, 1)$, let us consider the cone $C\left(O, -e_n, \arcsin\frac{R}{d}, \frac{d^2-R^2}{d}\right)$.

From now on, without loss of generality, we assume that

$$d_m(D_1, D_2) = \max_{x \in \partial D_1 \cap \partial\Omega_D} \mathrm{dist}(x, \partial D_2)$$

and we write $d_m = d_m(D_1, D_2)$.

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbb{R}^n \setminus \Omega_D$. Let us pick a point $P \in \partial D_1 \cap \partial\Omega_D$, let ν be the outer unit normal to ∂D_1 at P and let $d > 0$ be such that the segment $[(P + d\nu), P]$ is contained in $\mathbb{R}^n \setminus \Omega_D$. Given $P_0 \in \mathbb{R}^n \setminus \Omega_D$, let γ be a path in $\mathbb{R}^n \setminus \Omega_D$ joining P_0 to $P + d\nu$. We consider the following neighborhood of $\gamma \cup [(P + d\nu), P] \setminus \{P\}$ formed by a tubular neighborhood of γ attached to a cone with vertex at P and axis along ν

$$V(\gamma) = \bigcup_{S \in \gamma} B_R(S) \cup C\left(P, \nu, \arcsin\frac{R}{d}, \frac{d^2 - R^2}{d}\right). \tag{3}$$

Note that two significant parameters are associated to such a set, the radius R of the tubular neighborhood of γ, $\cup_{S\in\gamma} B_R(S)$, and the half-aperture $\arcsin\frac{R}{d}$ of the cone $C\left(P, \nu, \arcsin\frac{R}{d}, \frac{d^2-R^2}{d}\right)$. In other terms, $V(\gamma)$ depends on γ and also on the parameters R and d. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of R, d are introduced. Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P = O = (0,\dots,0)$ and $\nu = -e_n$.

Lemma 2. *Under the above notation, there exist positive constants $\overline{d}$, c_1, where $\frac{\overline{d}}{\rho_0}$ only depends on M_1 and α, and c_1 only depends on M_1, α, M_2, and there exists a point $P \in \partial D_1$ satisfying*

$$c_1 d_m \leq dist(P, D_2),$$

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset (\overline{\Omega^{\rho_0}} \cup S_{2\rho_0}) \setminus \overline{\Omega_D}$ joining P_0 to $P + \overline{d}\nu$, where ν is the unit outer normal to D_1 at P, such that, choosing a coordinate system with origin O at P and axis $e_n = -\nu$, the set $V(\gamma)$ introduced in (3) *satisfies*

$$V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D,$$

provided $R = \frac{\overline{d}}{\sqrt{1+L_0^2}}$, where L_0, $0 < L_0 \leq M_1$, is a constant only depending on M_1 and α.

Proof. See [Al-DC-Mo-Ro, Lamma 4.2].

A crucial tool to get the stability estimates is the so called Alessandrini identity [Al] the permits to relate the information provided by the boundary measurements with the unknown inclusion. Let $u_i \in H^1(\partial\Omega)$, $i = 1,2$, solutions to (1) with conductivities

$$\gamma_{D_i}(x) = c_1 A(x) + (c_2 - c_1) A(x) \chi_{\Omega_+} + (k - c_2) \chi_{D_i}, \qquad i = 1,2,$$

we have

$$\int_\Omega (\gamma_{D_1} \nabla u_1 \cdot \nabla u_2) - \int_\Omega (\gamma_{D_2} \nabla u_1 \cdot \nabla u_2) = \int_{\partial\Omega} c_1 A(x) u_1 [\Lambda_{D_1} - \Lambda_{D_2}] u_2. \tag{4}$$

Therefore, applying (4) replacing $u_i = \Gamma_{D_i}$, $i = 1,2$, where Γ_{D_i} is the fundamental solution of the operator $\mathrm{div}(\gamma_i \nabla \cdot)$, we get

$$\int_{D_1} (k - c_2) \nabla\Gamma_{D_1}(\cdot,y) \cdot \nabla\Gamma_{D_2}(\cdot,z) - \int_{D_2} (k - c_2) \nabla\Gamma_{D_1}(\cdot,y) \cdot \nabla\Gamma_{D_2}(\cdot,z)$$
$$= \int_{\partial\Omega} c_1 A(\cdot) \Gamma_{D_1}(\cdot,y) [\Lambda_{D_1} - \Lambda_{D_2}] \Gamma_{D_2}(\cdot,z). \tag{5}$$

For $y,z \in \mathscr{G} \cap \mathscr{C}\Omega$, where $\mathscr{C}\Omega$ is the complementary set of Ω, we define

$$S_{D_1}(y,z) = (k - c_2) \int_{D_1} \nabla\Gamma_{D_1}(\cdot,y) \cdot \nabla\Gamma_{D_2}(\cdot,z)$$

$$S_{D_2}(y,z) = (k - c_2) \int_{D_2} \nabla\Gamma_{D_1}(\cdot,y) \cdot \nabla\Gamma_{D_2}(\cdot,z)$$

$$f(y,z) = S_{D_1}(y,z) - S_{D_2}(y,z).$$

Therefore (5) can be written as

$$f(y,z) = \int_{\partial\Omega} c_1 A(\cdot)\Gamma_{D_1}(\cdot,y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(\cdot,z), \quad \forall y,z \in \mathscr{C}\overline{\Omega}. \tag{6}$$

In what follows, we analyze the behavior of f and S_{D_i} as the singularities y and z get close to the inclusion D.

Proposition 1. *Let Ω, D_1, D_2 be open sets satisfying the above properties and let $y = h\nu(O)$. If, given $\varepsilon > 0$, we have*

$$||\Lambda_{D_1} - \Lambda_{D_2}||_{L(H^{1/2},H^{-1/2})} < \varepsilon, \tag{7}$$

then for every h where $0 < h < cr, 0 < c < 1$, and c depends on M_1, we have

$$|f(y,y)| \leq C_0 \frac{\varepsilon^{Bh^F}}{h^T}. \tag{8}$$

Here $0 < T < 1$ and $C_0, B, F > 0$ are constants that depend only on the a priori data.

Proposition 2. *Let Ω, D_1, D_2 be open sets satisfying the above properties and let $y = h\nu(O)$. Then for every $0 < h < r_0/2$*

$$|S_{D_1}(y,y)| \geq C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3, \tag{9}$$

where $r_0 := \frac{r}{2} \min\left[\frac{1}{2}(8M_1)^{-1/\alpha}, \frac{1}{2}\right]$, and C_1, C_2, C_3 are positive constants depending only on the a priori data.

We can conclude this section proving our main theorem.

Proof (Proof of Theorem 1). We start from the origin of the coordinate system, point $O \in \partial D_1 \cap \partial\Omega_D$, for which the maximum in Definition 2 is attainted

$$d_m := d_m(D_1, D_2) = \mathrm{dist}(O, D_2).$$

By a transformation of coordinates, we can write $y = h\nu(O)$ where $0 < h < h_1, h_1 := \min\{d_m, cr, r_0/2\}, 0 < c < 1$, where c depends on M_1. By applying [Al-DC] Proposition 3.4 (i); i.e., $|\nabla_x \Gamma_{D_i}(x,y)| \leq d_1 |x-y|^{1-n}$, where $d_1 > 0$ depending only on k, n, α, M_1; we have

$$\begin{aligned} |S_{D_2}(y,y)| &= \left|(k-c_2)\int_{D_2} \nabla\Gamma_{D_1}(x,y)\nabla\Gamma_{D_2}(x,y)\right| \\ &\leq d_2(k-c_2)\int_{D_2} (d_1|x-y|^{1-n})^2 \leq d_1(k-c_2)d_1^2 \int_{D_2} (|d_m - h|^{1-n})^2 \\ &\leq d_1(k-c_2)d_1^2 |d_m - h|^{2-2n}|D_2| \leq C_4 |d_m - h|^{2-2n}, \end{aligned} \tag{10}$$

where d_2, C_4 are constants depending on the a priori data only. Here $|D_2|$ is the measure of the inclusion D_2 which is bounded by $|D_2| \leq |\Omega| \leq M_2 r_1^n$. If we apply the triangular inequality, we obtain

$$|S_{D_1}(y,y)|-|S_{D_2}(y,y)| \le |S_{D_1}(y,y)-S_{D_2}(y,y)| = |f(y,y)| \le C_0 \frac{\varepsilon^{Bh^F}}{h^T}. \tag{11}$$

Meanwhile, (9) gives us the lower bound of $S_{D_1}(y,y)$. Therefore, together with (10) and (11), we obtain

$$C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3 \le C_4 |d_m - h|^{2-2n} + C_0 \frac{\varepsilon^{Bh^F}}{h^T}$$

Rearranging terms we get

$$C_1 h^{2-n} \le C_4 |d_m - h|^{2-2n} + C_0 \frac{\varepsilon^{Bh^F}}{h^T}.$$

By setting $C_5 = C_4/C_0$ and $C_6 = C_1/C_0$

$$C_5 |d_m - h|^{2-2n} \ge C_6 h^{2-n} - \frac{\varepsilon^{Bh^F}}{h^T} = C_6 h^{2-n}(1 - \varepsilon^{Bh^F} h^K),$$

where $0 < K = n-2-T$. Now let $h = h(\varepsilon) = \min\left\{|\ln \varepsilon|^{-\frac{1}{2F}}, d_m\right\}$, for $0 < \varepsilon \le \varepsilon_1, \varepsilon_1 \in (0,1)$ such that $\exp(-B|\ln \varepsilon_1|^{1/2}) = 1/2$. It is easy to see if $d_m \le |\ln \varepsilon|^{-\frac{1}{2F}}$, Theorem 1 is proved using Lemma 1. Indeed we can set $\eta = \frac{1}{2F} > 0$, then

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \le c_0 d_m \le c_0 |\ln \varepsilon|^{-\eta} = \omega(\varepsilon) \tag{12}$$

In the other case if $d_m \ge |\ln \varepsilon|^{-\frac{1}{2F}}$, it is easy to check

$$(d_m - h)^{2-2n} \ge \frac{C_6}{2C_5} h^{2-n} \quad \Longrightarrow \quad d_m \le C_7 |\ln \varepsilon|^{-\frac{n-2}{4F(n-1)}}.$$

Here we can solve d_m because here $h = h(\varepsilon) = |\ln \varepsilon|^{-\frac{1}{2F}}$, and C_7 depends only on the *a priori* data. Therefore we conclude the proof by setting $\eta = \frac{n-2}{4F(n-1)}$

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \le c_0 d_m \le c_0 C_7 |\ln \varepsilon|^{-\eta} = \omega(\varepsilon) \tag{13}$$

and for $\varepsilon_1 \le \varepsilon$, we can also include the proof because $d_m \le |\Omega| \le M_2 r_1^n$.

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \le c_0 d_m \le c_0 M_2 r_1^n = \omega(\varepsilon). \tag{14}$$

We can conclude the proof Theorem 1 by (12), (13) and (14)

$$d_{\mathscr{H}}(\partial D_1, \partial D_2) \le C d_m = \omega(\varepsilon),$$

where C only depends on the *a priori* data.

4 Proof of Propositions 1 and 2

In this section we prove the auxiliary propositions needed to prove our main theorem. The proofs are based on some quantitative estimates of unique continaution, which for this special context has been developed in [DC-Fr-Li-Ve-Wa] (see also [Fr-Li-Ve-Wa].

Proof (Proposition 1). Let us consider $f(y,\cdot)$ with a fixed $y \in S_{2r}$ then

$$\mathrm{div}_w(\gamma_D(x)\nabla f(y,w)) = 0 \quad \text{in } \mathscr{C}\overline{\Omega}_D. \tag{15}$$

For $x \in S_{2r}$, by (6) and (7), we have the smallness quantity

$$|f(y,x)| \leq C(r,M_1,M_2)||\Gamma_{D_1} - \Gamma_{D_2}|| = \varepsilon. \tag{16}$$

Also by y [Al-DC] Proposition 3.4, the uniform bound of f is given as

$$|f(y,x)| \leq ch^{2-2n}, \qquad \text{in } \mathscr{G}^h \cup \mathscr{F}^\lambda. \tag{17}$$

At this point the proposition can be obtained using iteratively the three sphere inequality derived in [DC-Re3] for elliptic equation wtih coefficients with jump discontinuity (see also [Ca-Wa] for similar results) along the line of the proof of [Al-DC, Proposition 3.5].

Proof (Proposition 2). We write the upper bound of S_{D_1} as

$$\begin{aligned} |S_{D_1}(y,y)| &= \Big|(k-c_2)\int_{D_1} \nabla\Gamma_{D_1}(x,y)\nabla\Gamma_{D_2}(x,y)dx\Big| \\ &\geq C\Big|\Big(\int_{D_1\cap B_r(O)\cap D_2} + \int_{D_1\cap B_\rho(O)\cap\mathscr{C}D_2}\Big)\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \\ &-C\Big|\int_{D_1\cap B_r(O)\cap\mathscr{C}B_\rho(O)\cap\mathscr{C}D_2}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \\ &-C\Big|\int_{D_1\setminus B_r(O)}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \end{aligned} \tag{18}$$

where C depends on $k,\overline{A}$ only, $r = |x-y|, 0 < r < r_0, 0 < \rho < \min\{d_m, r\}$. To explain the formula, notice we separate the integrand $\int_{D_1\cap B_r(O)}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}$ into two parts, because we don't have any information on x. So, either it can be $x \in D_1 \cap B_r(O) \cap D_2$ or $x \in D_1 \cap B_r(O) \cap \mathscr{C}D_2$. Then we separate the integrand again with respect to an even smaller ball $B_\rho(O)$.

If $x \in D_1 \cap B_r(O) \cap D_2$, By [Al, Lemma 3.1] and [DC-Re, Theorem 4.1], we get

$$\nabla\Gamma_{D_1}(x,y)\cdot\nabla\Gamma_{D_2}(x,y) \geq C_A|x-y|^{2-2n} = C_A r^{2-2n} > 0 \tag{19}$$

where C_A depends on the *a priori* data. If $x \in D_1 \cap B_r(O) \cap \mathscr{C}D_2$, we consider in a smaller ball $B_\rho(O)$. In this case, we actually have $x \in D_1 \cap B_\rho(O) \cap \mathscr{C}D_2$. By definition of d_m, $B_\rho(O) \cap D_2 = \emptyset$, for $x, y \in B_\rho(O)$, we have

$$\begin{cases} \Delta\Big(\Gamma_{D_2}(x,y) - \Gamma(x,y)\Big) = 0 \quad \text{in} \;\; B_\rho(O) \\ \Big(\Gamma_{D_2}(x,y) - \Gamma(x,y)\Big)|_{\partial B_\rho(O)} \le C_K \rho^{2-n}, \end{cases}$$

where Γ denotes the standard fundamental solution of the Laplace operator. By the maximum principle, the value on interior is smaller than boundary

$$\Big|\Gamma_{D_2}(x,y) - \Gamma(x,y)\Big| \le C_K \rho^{2-n} \quad \forall x, y \in B_\rho(O)$$

And by interior gradient bound, we have

$$\Big|\nabla\Gamma_{D_2}(x,y) - \nabla\Gamma(x,y)\Big| \le C_{K_0} \rho^{1-n} \quad \forall x \in B_{\rho/2}(O); \forall y \in B_\rho(O)$$

Applying [A] Lemma 3.1 in $B_{\rho/2}(O)$, we have (notice $|x-y| = r > \rho$)

$$\nabla\Gamma_{D_1}(x,y) \cdot \nabla\Gamma_{D_2}(x,y) \ge C_A |x-y|^{2-2n} - C_K \rho^{2-2n} = C_A r^{2-2n} - C_K \rho^{2-2n} > 0 \quad (20)$$

Now we can bound the first term of (18) thanks to (19) and (20)

$$\begin{aligned} &\Big|\Big(\int_{D_1 \cap B_r(O) \cap D_2} + \int_{D_1 \cap B_\rho(O) \cap \mathscr{C} D_2}\Big) \nabla\Gamma_{D_1} \nabla\Gamma_{D_2}\Big| \\ \ge &\Big|\Big(\int_{D_1 \cap B_r(O) \cap D_2} + \int_{D_1 \cap B_\rho(O) \cap \mathscr{C} D_2}\Big)(C_A r^{2-2n} - C_K \rho^{2-2n})\Big| \\ \ge &\Big|\Big(\int_{[D_1 \cap B_r(O) \cap D_2] \cup [D_1 \cap B_\rho(O) \cap \mathscr{C} D_2]}\Big) c_1 r^{2-2n}\Big| \ge c_1 h^{2-n} \end{aligned} \quad (21)$$

For the upper bounds of the second and third term, we can apply the natural bound of $\nabla\Gamma_{D_i}, i = 1, 2$. When $x \in D_1 \cap B_r(O) \cap \mathscr{C} B_\rho(O) \cap \mathscr{C} D_2$, we have

$$\begin{aligned} &\Big|\int_{D_1 \cap B_r(O) \cap \mathscr{C} B_\rho(O) \cap \mathscr{C} D_2} \nabla\Gamma_{D_1} \nabla\Gamma_{D_2}\Big| \le \Big|\int_{D_1 \cap B_r(O) \cap \mathscr{C} B_\rho(O) \cap \mathscr{C} D_2} c_1 |x-y|^{1-n} \cdot c_1 |x-y|^{1-n}\Big| \\ \le &\Big|\int_{D_1 \cap B_r(O) \cap \mathscr{C} B_\rho(O) \cap \mathscr{C} D_2} c_1 r^{1-n} \cdot c_1 r^{1-n}\Big| \le c_2 d_m^{2-2n} \end{aligned} \quad (22)$$

$$\begin{aligned} \Big|\int_{D_1 \setminus B_r(O)} \nabla\Gamma_{D_1} \nabla\Gamma_{D_2}\Big| &\le \Big|\int_{D_1 \setminus B_r(O)} c_1 |x-y|^{1-n} \cdot c_1 |x-y|^{1-n} dx\Big| \\ &= \Big|\int_{D_1 \setminus B_r(O)} c_1^2 r^{2-2n} dx\Big| \\ &= c_3 \end{aligned} \quad (23)$$

Now we can plug (21), (22) and (23) into (18), we obtain the lower bound for $S_{D_1}(y,y)$

$$|S_{D_1}| \geq c_1 h^{2-n} - c_2 d_m^{2-2n} - c_3$$

where $c_i, i = 1,2,3$ depends only on the *a prior* data.

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