ATTRACTORS FOR THE HYPERBOLIC EQUATION

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The Model Equation

$$u_{tt} + \alpha u_t - \Delta u + \phi(u) = g, \qquad \alpha > 0$$

Conditions on ϕ and g. Let

$$g \in L^2$$

and let $\phi \in C(\mathbb{R})$ satisfying

$$|\phi(r)| \leq c(1 + |r|^3), \qquad \forall\, r \in \mathbb{R} \tag{H1}$$

and

$$\liminf_{|r|\to\infty} \frac{\phi(r)}{r} > -\lambda_0 \tag{H2}$$

where λ_0 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

Remark. There holds

$$\lambda_0 \|u\|^2 \leq \|\nabla u\|^2, \qquad \forall\, u \in H_0^1.$$

The phase space is

$$\mathcal{H} = H_0^1 \times L^2.$$

Linear Homogeneous Case

The equation is

$$u_{tt} + \alpha u_t - \Delta u = 0$$

For $\varepsilon \in [0, \varepsilon_0]$, with $\varepsilon_0 \leq \alpha$ to be determined later, consider the auxiliary variable $\xi = \partial_t u + \varepsilon u$. Multiply times ξ to get

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\xi\|^2\Big) + \varepsilon\|\nabla u\|^2 + (\alpha - \varepsilon)\|\xi\|^2 = \varepsilon(\alpha - \varepsilon)\langle u, \xi\rangle.$$

In the sequel set for simplicity $\alpha = 1$.

Absorbing Set

Lemma. *Let X be a Banach space, and let $\mathcal{Z} \subset C([0,\infty),X)$. Let $E : X \to \mathbb{R}$ be a function such that*

$$\sup_{t\in\mathbb{R}^+} E(z(t)) \geq -m, \qquad E(z(0)) \leq M,$$

for some $m, M \geq 0$ and every $z \in \mathcal{Z}$. In addition, assume that for every $z \in \mathcal{Z}$ the function $t \mapsto E(z(t))$ be continuously differentiable, and satisfy the differential inequality

$$\frac{d}{dt}E(z(t)) + \delta\|z(t)\|^2 \leq k \tag{1}$$

for some $\delta > 0$ and $k \geq 0$ independent of $z \in \mathcal{Z}$. Then for every $\varepsilon > 0$ there is $t_0 = \frac{m+M}{\varepsilon} > 0$ such that

$$E(z(t)) \leq \sup_{\zeta\in X}\Big\{E(\zeta) : \delta\|\zeta\|^2 \leq k+\varepsilon\Big\}, \qquad \forall\, t \geq t_0.$$

Proof. Up to adding the constant m to E, we may assume that $E \geq 0$ (and we redefine M accordingly). Notice first that, for a fixed t, the relation

$$\frac{d}{dt}E(z(t)) \geq -\varepsilon \tag{2}$$

implies

$$E(z(t)) \leq \sup_{\zeta\in X}\Big\{E(\zeta) : \delta\|\zeta\|^2 \leq k+\varepsilon\Big\}. \tag{3}$$

Indeed, if (2) holds, we get immediately from (1) that

$$\delta\|z(t)\|^2 \leq k+\varepsilon.$$

Set then $t_\varepsilon = M/\varepsilon$. Chosen $z \in \mathcal{Z}$, there is $t_0 \in [0, t_\varepsilon]$, depending on z, such that (2) holds for $t = t_0$. If not, we had

$$E(z(t_\varepsilon)) < -\varepsilon t_\varepsilon + E(z(0)) \leq -\varepsilon t_\varepsilon + M = 0$$

against the positivity of E. Let us define

$$t^* = \sup\Big\{\tau > t_0 :\ (3)\ \text{holds}\ \ \forall\, t \in [t_0,\tau]\Big\}.$$

We show that $t^* = \infty$, and, in particular, (3) holds for every $t \geq t_\varepsilon$, independently of $z \in \mathcal{Z}$. Indeed, if $t^* < \infty$, there is a sequence $t_n \downarrow t^*$ such that

$$E(z(t_n)) - E(z(t^*)) > 0$$

yielding

$$\frac{d}{dt}E(z(t^*)) \geq 0.$$

From the continuity of the derivative, there exists a right neighborhood J of t^* such that (2) holds for every $t \in J$. Hence (3) holds for every $t \in J$, in contradiction with the maximality of t^*. $\square$

Theorem. *There exist a positive constant K with the following property: given any $R \geq 0$ there is $t_0 = t_0(R) \geq 0$ such that, whenever*

$$\|z_0\|_{\mathcal{H}} \leq R$$

the inequality

$$\|S(t)z_0\|_{\mathcal{H}} \leq K$$

holds for every $t \geq t_0$.

Proof. Denote by $c > 0$ a generic constant. Let $z_0 = (u_0, u_1)$, and set

$$S(t)z_0 = z(t) = (u(t), \partial_t u(t)).$$

Notice that by (H2), there is $\nu > 0$ and $\rho > 0$ such that

$$r\phi(r) \geq -\lambda_0(1-\nu)r^2, \qquad |r| \geq \rho. \tag{4}$$

1. Introduce

$$F(r) = \int_0^r \phi(y)\,dy, \qquad r \in \mathbb{R}.$$

Since $F(u) \in L^1(\Omega)$ for every $u \in H_0^1$, we set

$$\mathcal{F}(u) = \int_\Omega F(u(x))\,dx, \qquad u \in H_0^1.$$

By (4)

$$2\mathcal{F}(\zeta_1) \geq -(1-\nu)\|\nabla\zeta_1\|^2 - c, \qquad \forall\, \zeta_1 \in H_0^1.$$

2. Fix $\varepsilon > 0$ small (to be determined). Define the function $E : \mathcal{H} \to \mathbb{R}$ as

$$E(\zeta) = \|\nabla\zeta_1\|^2 + \|\zeta_2 + \varepsilon\zeta_1\|^2 + 2\mathcal{F}(\zeta_1).$$

for $\zeta = (\zeta_1, \zeta_2) \in \mathcal{H}$. For ε small we get

$$E(\zeta) \geq \frac{\nu}{2}\|\zeta\|_{\mathcal{H}}^2 - c, \qquad \forall \zeta \in \mathcal{H}. \tag{5}$$

As a byproduct, E is bounded below. Moreover,

$$E(\zeta) \leq c\|\zeta\|_{\mathcal{H}}(1 + \|\zeta\|_{\mathcal{H}}^3), \qquad \forall\, \zeta \in \mathcal{H}. \tag{6}$$

In particular,

$$E(z(0)) = E(z_0) \le c\|z_0\|_{\mathcal{H}}\big(1 + \|z_0\|_{\mathcal{H}}^3\big) \le cR_0\big(1 + R_0^3\big).$$

3. Consider the auxiliary variable $\xi = \partial_t u + \varepsilon u$. Multiplying times ξ, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E(z) + \varepsilon\|\nabla u\|^2 + (1-\varepsilon)\|\xi\|^2\\ &= -\varepsilon\langle\phi(u), u\rangle + \varepsilon(1-\varepsilon)\langle u, \xi\rangle + \langle g, \xi\rangle.\end{aligned}$$

Estimates of RHS: By(4), for ε small,

$$-\varepsilon\langle\phi(u), u\rangle \le \frac{\nu}{2}\|\nabla u\|^2 + \varepsilon c.$$

Hence we get

$$\frac{d}{dt}E(z) + \varepsilon\|z\|^2 \le c.$$

Apply the Lemma and use (5)-(6). □

Remark. Build the Absorbing set $\mathcal{B}_0$.

Remark. For $z_0 \in \mathcal{B}_0$,

$$\|z(t)\|_{\mathcal{H}} \le C$$

Universal Attractor

Conditions on ϕ.

$$\phi(r) = kr^3 + \beta(r), \qquad k \ge 0$$

with

$$\beta'(r) \le c(1 + |r|^\gamma), \qquad \gamma \in [1, 2).$$

Exercise. Show that if $k > 0$ ϕ fulfills (H1)-(H2)

Decomposition. Decompose the solution $z = (u, \partial_t u)$ to with initial data $z_0 = (u_0, u_1) \in \mathcal{B}_0$ as

$$z = z_d + z_c = (v, \partial_t v) + (w, \partial_t w)$$

where

$$\begin{aligned}&\partial_{tt}v + \partial_t v - \Delta v + kv^3 = 0\\ &z_d(0) = z_0\end{aligned}$$

and

$$\begin{aligned}&\partial_{tt}w + \partial_t w - \Delta w + ku^3 - kv^3 + \beta(u) = g\\ &z_c(0) = 0.\end{aligned}$$

Lemma A. *z_d decays exponentially.*

Remark. Using Lemma A and the previous results, we have that

$$\|z(t)\|_{\mathcal{H}},\ \|z_c(t)\|_{\mathcal{H}},\ \|z_d\|_{\mathcal{H}} \le C$$

Lemma B. *For every $t>0$ $z_c \in \mathcal{K}(t)$, where $\mathcal{K}(t) \subset \mathcal{H}$ is compact.*

Proof. Let $A = -\Delta$. Fix $t > 0$ and let $c = c(t)$ a generic constant. Multiply the equation times $A^s w_t$ (s to be fixed) to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|A^{(1+s)/2}w\|^2 + \|A^{s/2}w_t\|^2\Big) + \|A^{s/2}w_t\|^2 \\ = -k\langle(u^3-v^3), A^s w_t\rangle + \langle\beta(u), A^s w_t\rangle + \langle g, A^s w_t\rangle\end{aligned}$$

Remark. We have, for $s \in [0, 1/2)$

$$\begin{aligned}\mathcal{D}(A^{s/2}) &\hookrightarrow H^s \hookrightarrow L^{6/(3-2s)}\\ \mathcal{D}(A^{(1+s)/2}) &\hookrightarrow H^{1+s} \hookrightarrow L^{6/(1-2s)}\\ \mathcal{D}(A^{(1-s)/2}) &\hookrightarrow H^{1-s} \hookrightarrow L^{6/(1+2s)}\end{aligned}$$

Integrate on $(0,t)$:

$$\begin{aligned}\int_0^t \langle\beta(u), A^s w_t\rangle &= \langle\beta(u), A^s w\rangle - \int_0^t \langle\beta'(u)u_t, A^s w\rangle\\ &\le c\|\beta(u)\|\|A^s w\| + c\int_0^t \|\beta'(u)\|_{6/\gamma}\|u_t\|_2\|A^s w\|_{6/3-\gamma}\end{aligned}$$

and fix $s > 0$ such that $6/(3-\gamma) \le 6/(1+2s)$ (possible iff $\gamma < 2$).

$$\int_0^t \langle g, A^s w_t\rangle = \langle g, A^s w\rangle \le c\|g\|\|A^s w\|$$

$$-k\int_0^t \langle(u^3-v^3), A^s w_t\rangle = -k\langle(u^3-v^3), A^s w\rangle + 3k\int_0^t \langle(u^2u_t - v^2v_t), A^s w\rangle$$

$$\int_0^t \langle(u^2u_t - v^2v_t), A^s w\rangle = \int_0^t \langle(u+v)u_t w, A^s w\rangle - \int_0^t \langle v^2 w_t, A^s w\rangle$$

But

$$\int_0^t \langle(u+v)u_t w, A^s w\rangle \le c\int_0^t \|u+v\|_6\|u_t\|_2\|w\|_{6/(1-2s)}\|A^s w\|_{6/(1+2s)}$$

and

$$-\int_0^t \langle v^2 w_t, A^s w\rangle \le c\int_0^t \|v^2\|_3\|w_t\|_{6/(3-2s)}\|A^s w\|_{6/(1+2s)}$$

Collect everything, to get

$$\Phi(t) \le c + c\int_0^t \Phi.$$