1. Unbounded operators

Let H be a separable Hilbert space. We consider linear operators

$$A : \mathcal{D}(A) \subset H \to H$$

with $\mathcal{D}(A)$ dense subspace of H.

Consider for instance the operator $A = d/dx$ in the domain $C^1([0,1]) \subset L^2(0,1)$. Choosing $u_k(x) = e^{kx}$ we see at once that the operator is not bounded. Indeed,

$$\|Au_k\|_{L^2} = k\|u_k\|_{L^2}$$

hence

$$\sup \left\{ \|Au\|_{L^2} : \, u \in C^1([0,1]), \, \|u\|_{L^2} = 1 \right\} = \infty.$$

So operators not defined everywhere are expected to be unbounded.

We are actually interested to a special class of operators, namely, *closed* operators.

> **Definition** A is closed if for all $x_n \in \mathcal{D}(A)$, $x_n \to x$ with $Ax_n \to y$, then $x \in \mathcal{D}(A)$ and $Ax = y$.

Here is of a certain importance the domain of A. Indeed, A might not be closed, but it can have a closed extension.

It is well-known that when $\mathcal{D}(A) = H$, if A is closed then A is continuous [Closed Graph Theorem].

> **Remark** Here is a remarkable consequence of the Closed Graph Theorem (or, if you like, of the Inverse Mapping Theorem):
> *If X is a Banach space with respect two two different norms $\|\cdot\|$ and $\|\cdot\|_*$, and, for some $K < \infty$,*
>
> $$\|x\| \leq K\|x\|_*, \qquad \forall \, x \in X$$
>
> *then there is $k < \infty$ such that*
>
> $$\|x\| \geq k\|x\|_*, \qquad \forall \, x \in X$$
>
> *namely, the two norms are equivalent* [**Exercise**].
>
> In general we have
>
> **Theorem** *Let A be closed. Then A is bounded if and only if $\mathcal{D}(A) = H$.*

Definition A is *symmetric* if

$$\langle Ax, y\rangle = \langle x, Ay\rangle, \qquad \forall\, x, y \in \mathcal{D}(A).$$

Symmetric operators are always closable, and the closure is still symmetric; hence we will implicitly consider the closure of a symmetric operator.

1.1 The bounded case Let us consider for a while the case $\mathcal{D}(A) = H$. Then a symmetric operator is self-adjoint. It is well-known that bounded self-adjoint operators have a spectral decomposition. In particular we are interested to a special class of linear bounded operators, namely, *compact operators.*

Theorem *Let K be a symmetric compact operator. Then all the eigenvalues of K are real, and eigenspaces of nonzero eigenvalues are finite-dimensional. If the nonzero eigenvalues λ_j (counted with multiplicity) are ordered so that $|\lambda_{j+1}| \le |\lambda_j|$, then*

$$\lim_j \lambda_j = 0.$$

Moreover the (normalized) eigenvectors w_j of λ_j form an orthonormal family of H. This family is complete if and only if 0 is not an eigenvalue. Hence we have the spectral decomposition

$$Kx = \sum_j \lambda_j \langle x, w_j\rangle w_j.$$

The converse of this result holds too.

Theorem *Let λ_j be a real sequence converging to zero, let ω_j be an orthonormal family of H, and let $K : H \to H$ be defined as*

$$Kx = \sum_j \lambda_j \langle x, w_j\rangle w_j.$$

operator. Then K is a symmetric compact operator.

In the unbounded case the notion of self-adjoint operator is more involved.

1.2 The adjoint operator *Given A, the domain of adjoint operator A^* of A consists of all $y \in H$ for which the linear functional*

$$x \to \langle Ax, y\rangle$$

is continuous on $\mathcal{D}(A)$. If $y \in \mathcal{D}(A^*)$, then the Bounded Extension Theorem uniquely extends the above functional to a functional on H. Therefore there is a unique element $A^*y \in H$ that satisfies

$$\langle Ax, y\rangle = \langle x, A^*y\rangle, \qquad \forall\, x \in \mathcal{D}(A).$$

Standard verifications show that $\mathcal{D}(A^*)$ is a subspace [**Exercise**]. *Of course it might happen that $\mathcal{D}(A^*)$ is no longer densely defined. However, if A is closed then $\mathcal{D}(A^*)$ is dense in H. Moreover A^* is closed and $A^{**} = A$ (Notice: two unbounded operators A and B are equal if $\mathcal{D}(A) = \mathcal{D}(B)$, and $Ax = Bx$ for all x in the domain).*

Definition *A is self-adjoint if $A = A^*$.*

It is clear that, in general, it is not an easy task to verify whether an operator is self-adjoint or not. The following criterion turns out to be quite useful:

Theorem *Let A be a symmetric operator.*

1. *If $\mathcal{D}(A) = H$ then A is self-adjoint and $A \in L(H)$.*

2. *If* Range$(A) = H$ *then A is self-adjoint and one-to-one. In addition, its inverse $A^{-1} \in L(H)$ and $A^{-1} = (A^{-1})^*$.*

The spectral decomposition we need is the following.

1.3 Theorem *Let A be a self-adjoint operator. Assume that A is invertible and A^{-1} is compact. Then all the eigenvalues of A are real and different from zero, and corresponding eigenspaces are finite-dimensional. If the eigenvalues α_j (counted with multiplicity) are ordered so that $|\alpha_{j+1}| \geq |\alpha_j|$, then*

$$\lim_j \alpha_j = \infty.$$

The corresponding (normalized) eigenvectors w_j of α_j form a complete orthonormal family of H. That is, we have the spectral decomposition

$$Ax = \sum_j \alpha_j \langle x, w_j \rangle w_j.$$

Moreover,

$$A^{-1} w_j = \frac{1}{\alpha_j} w_j$$

1.4 Positive operators Let A be a self-adjoint operator. A is (strictly) positive if

$$\langle Ax, x \rangle \geq k \|x\|^2, \qquad \forall\, x \in \mathcal{D}(A)$$

for some $k > 0$. If we are in the hypotheses of Theorem 1.3, all the eigenvalues are positive. We can then use this fact to define *fractional powers* of A, by

$$A^s x = \sum_j \alpha_j^s \langle x, w_j \rangle w_j, \qquad s \in \mathbb{R}.$$

The domain of A^s is made of those $x \in H$ for which the above series converges, namely,

$$\mathcal{D}(A^s) = \Big\{ x \in H : \sum_j \alpha_j^{2s} |\langle x, w_j \rangle|^2 < \infty \Big\}$$

which is to say, loosely speaking,

$$\mathcal{D}(A^s) = \Big\{ x \in H : A^s x \in H \Big\}.$$

Let us now examine some issues about A^s.

(i) For $s > 0$, A^{-s} is a compact operator. In particular, $\mathcal{D}(A^{-s}) = H$. Moreover, $\mathrm{Range}(A^{-s}) = \mathcal{D}(A^s)$. Indeed,

$$y = \sum_j c_j w_j \in \mathcal{D}(A^s)$$

if and only if

$$x = \sum_j \alpha_j^s c_j w_j \in H.$$

In this case it follows at once that $A^{-s} x = y$.

(ii) For $s > 0$, A^s is a (strictly) positive operator. Thus $\mathcal{D}(A^s)$ is dense in H. Moreover, A^s is invertible and $(A^s)^{-1} = A^{-s}$.

(iii) For $s, r \in \mathbb{R}$, $A^s A^r = A^r A^s = A^{s+r}$ in the common domain of definition of the three operators.

(iv) For $s \in \mathbb{R}$, $\mathcal{D}(A^s)$ is a Hilbert space with the product

$$\langle x, y \rangle_{\mathcal{D}(A^s)} = \langle A^s x, A^s y \rangle.$$

and norm

$$\|x\|_{\mathcal{D}(A^s)} = \|A^s x\|.$$

The completeness of $\mathcal{D}(A^s)$ is immediate, since the operator A^s maps $\mathcal{D}(A^s)$ isometrically isomorphically onto H. The vectors

$$\tilde{w}_j = \frac{1}{\alpha_j^s} w_j$$

form a complete orthonormal family of $\mathcal{D}(A^s)$.

(v) When $s \geq r$, the embedding $\mathcal{D}(A^s) \subset \mathcal{D}(A^r)$ is continuous and dense. When $s > r \geq 0$ the embedding $\mathcal{D}(A^s) \subset \mathcal{D}(A^r)$ is also compact.

2. Inequalities

2.1 Young Inequality *Let* $a, b \geq 0$, $p, q > 1$ *with* $1/p + 1/q = 1$. *Then for* $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q$$

with $C(\varepsilon) = (\varepsilon p)^{-q/p}/q$.

2.2 Generalized Hölder Inequality *Let* $p_1, \dots, p_n$ *be such that* $1 \leq p_j \leq \infty$ *with* $\sum 1/p_j = 1$. *Then, if* $f_j \in L^{p_j}(\Omega)$,

$$\int_\Omega |f_1 \dots f_n| \, dx \leq \prod_{j=1}^n \|f_j\|_{L^{p_j}(\Omega)}.$$

2.3 Gronwall Lemma [Differential form] *Let* η *be an absolutely continuous function on* $[t_0, T]$ *and* ϕ, ψ *two summable functions on* $[t_0, T]$ *which satisfy the differential inequality*

$$\frac{d}{dt}\eta(t) \leq \phi(t)\eta(t) + \psi(t)$$

for a.e. $t \in [t_0, T]$. *Then*

$$\eta(t) \leq \eta(t_0) e^{\int_{t_0}^t \phi(s)\, ds} + \int_{t_0}^t e^{\int_s^t \phi(r)\, dr} \psi(s) \, ds$$

for all $t \in [t_0, T]$.

PROOF We have

$$\begin{aligned} \frac{d}{ds}\Big(\eta(s) e^{-\int_{t_0}^s \phi(r)\, dr}\Big) &= e^{-\int_{t_0}^s \phi(r)\, dr}\Big(\eta'(s) - \phi(s)\eta(s)\Big) \\ &\leq e^{-\int_{t_0}^s \phi(r)\, dr}\psi(s). \end{aligned}$$

Integrate then in s on $[t_0, t]$, for any $t \in [t_0, T]$. ◇

Remark Here are some particular instances of a certain importance:

- If $\psi \equiv 0$ and $\eta(t_0) = 0$ then $\eta \leq 0$. This is of particular interest when one knows in advance that η is positive.

- If ϕ, ψ are nonnegative, we get
$$\eta(t) \leq e^{\int_{t_0}^t \phi(s)\,ds}\left[\eta(t_0) + \int_{t_0}^t \psi(s)\,ds\right].$$
- If $\phi = -\varepsilon$, $\varepsilon \geq 0$, then
$$\eta(t) \leq \eta(t_0) e^{-\varepsilon(t-t_0)} + \int_{t_0}^t e^{-\varepsilon(t-s)}\psi(s)\,ds.$$
- If $\phi = -\varepsilon$, $\varepsilon \geq 0$, and $\psi = k \geq 0$, then
$$\eta(t) \leq \eta(t_0) e^{-\varepsilon(t-t_0)} + \frac{k}{\varepsilon}.$$
Notice that this estimate is "good" as $t \to \infty$.

Corollary Let η be an absolutely continuous positive function on $[t_0, T]$ and ϕ, f, g three positive summable functions on $[t_0, T]$ which satisfy the differential inequality
$$\frac{d}{dt}\eta(t) \leq \phi(t)\eta(t) + f(t) + g(t)\sqrt{\eta(t)}$$
for a.e. $t \in [t_0, T]$. Then
$$\eta(t) \leq 2e^{\int_{t_0}^t \phi(s)\,ds}\left(\eta(t_0) + \int_{t_0}^t f(s)\,ds\right) + e^{2\int_{t_0}^t \phi(s)\,ds}\left(\int_{t_0}^t g(s)\,ds\right)^2$$
for all $t \in [t_0, T]$.

PROOF Set
$$\Phi(t) = \int_{t_0}^t \phi(s)\,ds, \qquad F(t) = \int_{t_0}^t f(s)\,ds, \qquad G(t) = \int_{t_0}^t g(s)\,ds.$$
Notice that Φ, F, G are positive and increasing. For a fixed $t \in [t_0, T]$, the Gronwall Lemma entails
$$\eta(t) \leq e^{\Phi(t)}\big[\eta(t_0) + F(t)\big] + e^{\Phi(t)}\int_{t_0}^t g(s)\sqrt{\eta(s)}\,ds.$$
Let now $t^* \in [t_0, t]$ be such that
$$\eta(t^*) = \max_{s\in[t_0,t]} \eta(s).$$
Then, from the Young inequality,
$$\begin{aligned} \eta(t^*) &\leq e^{\Phi(t)}\big[\eta(t_0) + F(t)\big] + \sqrt{\eta(t^*)}\, e^{\Phi(t)}G(t) \\ &\leq e^{\Phi(t)}\big[\eta(t_0) + F(t)\big] + \frac{\eta(t^*)}{2} + \frac{e^{2\Phi(t)}(G(t))^2}{2} \end{aligned}$$
which yields
$$\eta(t^*) \leq 2e^{\Phi(t)}\big[\eta(t_0) + F(t)\big] + e^{2\Phi(t)}(G(t))^2.$$
Since $\eta(t) \leq \eta(t^*)$ we are done. ◇

2.4 Gronwall Lemma [Integral form] *Let f be a continuous function on $[t_0,T]$ and h be a summable nonnegative function on $[t_0,T]$ which satisfy the integral inequality*

$$f(t) \le K + \int_{t_0}^{t} h(s) f(s)\, ds$$

for all $t \in [t_0,T]$ and some $K \in \mathbb{R}$. Then

$$f(t) \le K e^{\int_{t_0}^{t} h(s)\, ds}$$

for all $t \in [t_0,T]$.

PROOF Setting

$$\eta(t) = \int_{t_0}^{t} h(s) f(s)$$

we have

$$\eta'(t) = h(t) f(t) \le \eta(t) h(t) + K h(t)$$

since $h \ge 0$. Applying the Gronwall Lemma (differential form), we are led to

$$\eta(t) \le \int_{t_0}^{t} e^{\int_s^t h(r)\, dr} K h(s)\, ds.$$

But

$$f(t) \le K + \eta(t).$$

Hence,

$$f(t) \le K\Big(1 + \int_{t_0}^{t} e^{\int_s^t h(r)\, dr} h(s)\, ds\Big).$$

Now notice that

$$\frac{d}{ds} e^{\int_s^t h(r)\, dr} = -e^{\int_s^t h(r)\, dr} h(s)$$

which leads to

$$\int_{t_0}^{t} e^{\int_s^t h(r)\, dr} h(s)\, ds = -\int_{t_0}^{t} \frac{d}{ds} e^{\int_s^t h(r)\, dr}\, ds = -1 + e^{\int_{t_0}^{t} h(r)\, dr}.$$

Therefore the result follows. ◇

2.5 Uniform Gronwall Lemma *Let η be an absolutely continuous positive function on $[t_0,\infty)$ and ϕ, ψ two positive locally summable functions on $[t_0,\infty)$ which satisfy*

$$\frac{d}{dt}\eta(t) \le \phi(t)\eta(t) + \psi(t), \qquad a.e.\ t \ge t_0$$

$$\int_t^{t+r} \phi(s)\, ds \le a_1, \quad \int_t^{t+r} \psi(s)\, ds \le a_2, \quad \int_t^{t+r} \eta(s)\, ds \le a_3, \qquad \forall\, t \ge t_0$$

for some r, a_j *positive. Then*

$$\eta(t+r) \le \Big(\frac{a_3}{r} + a_2\Big) e^{a_1}, \qquad \forall\, t \ge t_0.$$

PROOF Let $t_0 \le t \le s \le t+r$. Then

$$\frac{d}{ds}\left(\eta(s) e^{-\int_t^s \phi(r)\,dr}\right) = e^{-\int_t^s \phi(r)\,dr}\Big(\eta'(s) - \phi(s)\eta(s)\Big) \le e^{-\int_t^s \phi(r)\,dr}\psi(s) \le \psi(s).$$

Integrate then in s on $[z, t+r]$ for $z > t$:

$$\eta(t+r) \le \big(\eta(z) + a_2\big) e^{a_1}.$$

Finally, integrate in z on $[t, t+r]$. ◇