Positive operator valued measures covariant with respect to an irreducible representation

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Abstract

Given an irreducible representation of a group G, we show that all the covariant positive operator valued measures based on G/Z, where Z is a central subgroup, are described by trace class, trace one positive operators.

1 Introduction

It is well known [2, 6] that, given a square-integrable representation π of a unimodular group G and a trace class, trace one positive operator T, the family of operators

$$Q(X) = \int_X \pi(g) T \pi(g^{-1}) d\mu_G(g),$$

defines a positive operator valued measure (POVM) on G covariant with respect to π (μ_G is a Haar measure on G). In this paper, we prove that all the covariant POVMs are of the above form for some T. More precisely, we show this result for non-unimodular groups and for POVMs based on the quotient space G/Z, where Z is a central subgroup.

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Let G be a locally compact second countable topological group and Z be a central closed subgroup. We denote by G/Z the quotient group and by $\dot{g} \in G/Z$ the equivalence class of $g \in G$. If $a \in G$ and $\dot{g} \in G/Z$, we let $a[\dot{g}] = \dot{a}\dot{g}$ be the natural action of a on the point $\dot{g}$.

Let $\mathcal{B}(G/Z)$ be the Borel σ-algebra of G/Z. We fix a left Haar measure $\mu_{G/Z}$ on G/Z. Moreover, we denote by Δ the modular function of G and of G/Z.

By *representation* we mean a strongly continuous unitary representation of G acting on a complex and separable Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$ linear in the first argument.

Let $(\pi, \mathcal{H})$ be a representation of G. A positive operator valued measure Q defined on G/Z and such that

1. $Q(G/Z) = I$;
2. for all $X \in \mathcal{B}(G/Z)$,
$$\pi(g) Q(X) \pi\left(g^{-1}\right) = Q(g[X]) \qquad \forall g \in G$$

is called π-*covariant* POVM on G/Z.

Given a representation $(\sigma, \mathcal{K})$ of Z, we denote by $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ the imprimitivity system unitarily induced by σ. We recall that $\mathcal{H}^\sigma$ is the Hilbert space of (μ_G-equivalence classes of) functions $f : G \longrightarrow \mathcal{K}$ such that

1. f is weakly measurable;
2. for all $z \in Z$,
$$f(gz) = \sigma\left(z^{-1}\right) f(g) \qquad \forall g \in G;$$
3. $$\int_{G/Z} \|f(g)\|_{\mathcal{K}}^2 \, \mathrm{d}\mu_{G/Z}(\dot{g}) < +\infty$$

with scalar product

$$\langle f_1, f_2 \rangle_{\mathcal{H}^\sigma} = \int_{G/Z} \langle f_1(g), f_2(g) \rangle_{\mathcal{K}} \, \mathrm{d}\mu_{G/Z}(\dot{g}).$$

The representation λ^σ acts on $\mathcal{H}^\sigma$ as

$$(\lambda^\sigma(a) f)(g) := f\left(a^{-1} g\right) \qquad g \in G$$

for all $a \in G$. The projection valued measure P^σ is given by

$$(P^\sigma(X) f)(g) := \chi_X(\dot{g}) f(g) \qquad g \in G.$$

for all $X \in \mathcal{B}(G/Z)$, where χ_X is the characteristic function of the set X.

We recall some basic properties of square integrable representations modulo a central subgroup. We refer to Ref. [1] for G unimodular and Z arbitrary and to Ref. [4] for G non-unimodular and $Z = \{e\}$. Combining these proofs, one obtains the following result.

Proposition 1 *Let $(\pi, \mathcal{H})$ be an irreducible representation of G and γ be the character of Z such that*

$$\pi(z) = \gamma(z) I_{\mathcal{H}} \qquad \forall z \in Z.$$

The following facts are equivalent:

1. *there exists a vector $u \in \mathcal{H}$ such that*

$$0 < \int_{G/Z} |\langle u, \pi(g) u\rangle_{\mathcal{H}}|^2 \, d\mu_{G/Z}(\dot{g}) < +\infty; \tag{1}$$

2. *$(\pi, \mathcal{H})$ is a subrepresentation of $(\lambda^\gamma, \mathcal{H}^\gamma)$.*

If any of the above conditions is satisfied, there exists a selfadjoint injective positive operator C with dense range such that

$$\pi(g) C = \Delta(g)^{-\frac{1}{2}} C\pi(g) \qquad \forall g \in G,$$

and an isometry $\Sigma : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H}^\gamma$ such that

1. *for all $u \in \mathcal{H}$ and $v \in \operatorname{dom} C$*

$$\Sigma(u \otimes v^*)(g) = \langle u, \pi(g) Cv\rangle_{\mathcal{H}} \qquad g \in G,$$

2. *for all $g \in G$*

$$\Sigma(\pi(g) \otimes I_{\mathcal{H}^*}) = \lambda(g)\Sigma,$$

3. *the range of Σ is the isotypic space of π in $\mathcal{H}^\gamma$.*

If Eq. (1) is satisfied, $(\pi, \mathcal{H})$ is called *square-integrable modulo Z*. The square root of C is called *formal degree* of π (see Ref. [4]). In particular, when G is unimodular, C is a multiple of the identity.

2 Characterization of Q

We fix an irreducible representation $(\pi, \mathcal{H})$ of G and let γ be the character such that $\pi|_Z = \gamma I_{\mathcal{H}}$. The following theorem characterizes all the POVM on G/Z covariant with respect to π in terms of positive trace one operators on $\mathcal{H}$.

Theorem 2 *The irreducible representation π admits a covariant POVM based on G/Z if and only if π is square-integrable modulo Z.*

In this case, let C be the square root of the formal degree of π. There exists a one-to-one correspondence between covariant POVMs Q on G/Z and positive trace one operators T on $\mathcal{H}$ given by

$$\langle Q_T(X) v, u\rangle_{\mathcal{H}} = \int_X \left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u\right\rangle_{\mathcal{H}} d\mu_{G/Z}(\dot{g}) \qquad (2)$$

for all $u, v \in \operatorname{dom} C$ and $X \in \mathcal{B}(G/Z)$.

Proof. Let Q be a π-covariant POVM. According to the generalized imprimitivity theorem [3] there exists a representation $(\sigma, \mathcal{K})$ of Z and an isometry $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$ intertwining π with λ^σ such that

$$Q(X) = W^* P^\sigma(X) W$$

for all $X \in \mathcal{B}(G/Z)$.

Define the following closed invariant subspace of $\mathcal{K}$

$$\mathcal{K}_\gamma = \{v \in \mathcal{K} \mid \sigma(z) v = \gamma(z) v\}.$$

Let σ_1 and σ_2 be the restrictions of σ to $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^\perp$ respectively. The induced imprimitivity system $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ decomposes into the orthogonal sum

$$\mathcal{H}^\sigma = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_2}.$$

If $f \in \mathcal{H}^\sigma$ and $z \in Z$, then

$$(\lambda^\sigma(z) f)(g) = f\left(z^{-1} g\right) = f\left(g z^{-1}\right) = \sigma(z) f(g) \qquad g \in G.$$

On the other hand, if $u \in \mathcal{H}$ and $z \in Z$, we have

$$(\lambda^\sigma(z) W u)(g) = (W\pi(z) u)(g) = \gamma(z)(W u)(g) \qquad g \in G.$$

It follows that $(Wu)(g) \in \mathcal{K}_\gamma$ for μ_G-almost every $g \in G$, that is, $Wu \in \mathcal{H}^{\sigma_1}$. So it is not restrictive to assume that

$$\sigma = \gamma I_{\mathcal{K}}$$

for some Hilbert space $\mathcal{K}$. Clearly, we have

$$\mathcal{H}^\sigma = \mathcal{H}^\gamma \otimes \mathcal{K}, \qquad \lambda^\sigma = \lambda^\gamma \otimes I_\mathcal{K}.$$

In particular, π is a subrepresentation of λ^γ, hence it is square-integrable modulo Z.

Due to Prop. 1, the operator $W' = (\Sigma^* \otimes I_\mathcal{K}) W$ is an isometry from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{K}$ such that

$$W'\pi(g) = \pi(g) \otimes I_{\mathcal{H}^* \otimes \mathcal{K}} \qquad g \in G.$$

Since π is irreducible, there is a unit vector $B \in \mathcal{H}^* \otimes \mathcal{K}$ such that

$$W'u = u \otimes B \qquad \forall u \in \mathcal{H}.$$

Let $(e_i)_{i \geq 1}$ be an orthonormal basis of $\mathcal{H}$ such that $e_i \in \operatorname{dom} C$, then

$$B = \sum e_i^* \otimes k_i,$$

where $k_i \in \mathcal{K}$ and $\sum_i \|k_i\|_\mathcal{K}^2 = 1$.

If $u \in \operatorname{dom} C$, one has that

$$\begin{aligned}
(Wu)(g) &= \left[(\Sigma \otimes I_\mathcal{K})\,(u \otimes B)\right](g) \\
&= \sum\nolimits_i \Sigma(u \otimes e_i^*)(g) \otimes k_i \\
&= \sum\nolimits_i \langle u, \pi\,(g)\, C e_i\rangle_\mathcal{H} \otimes k_i \\
&= \sum\nolimits_i \left\langle C\pi\left(g^{-1}\right) u, e_i\right\rangle_\mathcal{H} \otimes k_i \\
&= \sum\nolimits_i (e_i^* \otimes k_i)(C\pi\left(g^{-1}\right) u),
\end{aligned}$$

where the series converges in $\mathcal{H}^\sigma$. On the other hand, for all $g \in G$ the series $\sum_i (e_i^* \otimes k_i)(C\pi\left(g^{-1}\right) u)$ converges to $BC\pi\left(g^{-1}\right) u$, where we identify $\mathcal{H}^* \otimes \mathcal{K}$ with the space of Hilbert-Schmidt operators. By unicity of the limit

$$(Wu)(g) = BC\pi\left(g^{-1}\right) u \qquad g \in G.$$

If $u, v \in \operatorname{dom} C$, the corresponding covariant POVM is given by

$$\begin{aligned}
\langle Q\,(X)\, v, u\rangle_\mathcal{H} &= \langle P^\sigma\,(X)\, Wv, Wu\rangle_{\mathcal{H}^\sigma} \\
&= \int_{G/Z} \chi_X\,(\dot{g}) \left\langle BC\pi\left(g^{-1}\right) v, BC\pi\left(g^{-1}\right) u\right\rangle_\mathcal{H} \mathrm{d}\mu_{G/Z}\,(\dot{g}) \\
&= \int_X \left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u\right\rangle_\mathcal{H} \mathrm{d}\mu_{G/Z}\,(\dot{g}),
\end{aligned}$$

where

$$T := B^* B$$

is a positive trace class trace one operator on $\mathcal{H}$.

Conversely, assume that π is square integrable and let T be a positive trace class trace one operator on $\mathcal{H}$. Then

$$B := \sqrt{T}$$

is a (positive) operator belonging to $\mathcal{H}^* \otimes \mathcal{H}$ such that $B^* B = T$ and $\|B\|_{\mathcal{H}^* \otimes \mathcal{H}} = 1$. The operator W defined by

$$Wv := (\Sigma \otimes I_{\mathcal{H}})(v \otimes B) \qquad \forall v \in \mathcal{H}$$

is an isometry intertwining $(\pi, \mathcal{H})$ with the representation $(\lambda^\sigma, \mathcal{H}^\sigma)$, where

$$\sigma = \gamma I_{\mathcal{H}}.$$

Define Q_T by

$$Q_T(X) = W^* P^\sigma(X) W \qquad X \in \mathcal{B}(G/Z).$$

With the same computation as above, one has that

$$\langle Q_T(X) u, v \rangle_{\mathcal{H}} = \int_X \left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u \right\rangle_{\mathcal{H}} \mathrm{d}\mu_{G/Z}(\dot{g})$$

for all $u, v \in \operatorname{dom} C$.

Finally, we show that the correspondence $T \longmapsto Q_T$ is injective. Let T_1 and T_2 be positive trace one operators on $\mathcal{H}$, with $Q_{T_1} = Q_{T_2}$. Set $T = T_1 - T_2$. Since π is strongly continuous, for all $u, v \in \operatorname{dom} C$ the map

$$\begin{aligned} G/Z \ni \dot{g} \longmapsto & \left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u \right\rangle_{\mathcal{H}} \\ & = \Delta(\dot{g})^{-1} \left\langle T\pi\left(g^{-1}\right) Cv, \pi\left(g^{-1}\right) Cu \right\rangle_{\mathcal{H}} \in \mathbb{C} \end{aligned}$$

is continuous. Since

$$\int_X \left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u \right\rangle_{\mathcal{H}} \mathrm{d}\mu_{G/Z}(\dot{g}) = \langle [Q_{T_1}(X) - Q_{T_2}(X)] v, u \rangle_{\mathcal{H}} = 0$$

for all $X \in \mathcal{B}(G/Z)$, we have

$$\left\langle TC\pi\left(g^{-1}\right) v, C\pi\left(g^{-1}\right) u \right\rangle_{\mathcal{H}} = 0 \qquad \forall \dot{g} \in G/Z.$$

In particular,

$$\langle TCv, Cu \rangle_{\mathcal{H}} = 0,$$

so that, since C has dense range, $T = 0$. ∎

Remark 3 *Scutaru shows in Ref. [6] that there exists a one-to-one correspondence between positive trace one operators on $\mathcal{H}$ and covariant POVMs Q based on G/Z with the property*

$$\operatorname{tr} Q(K) < +\infty \tag{3}$$

for all compact sets $K \subset G/Z$. Theorem 2 shows that every *covariant POVM Q based on G/Z shares property (3).*

Remark 4 *If G is unimodular, then $K = \lambda I$, with $\lambda > 0$, and one can normalize $\mu_{G/Z}$ so that $\lambda = 1$. Hence,*

$$Q_T(X) = \int_X \pi(g) T \pi\left(g^{-1}\right) d\mu_{G/Z}(\dot{g}) \qquad \forall X \in \mathcal{B}(G/Z),$$

the integral being understood in the weak sense.

Remark 5 *If $T = \eta^* \otimes \eta$, with $\eta \in \operatorname{dom} C$ and $\|\eta\|_{\mathcal{H}} = 1$, we observe that*

$$\begin{aligned}
\langle Q_T(X) v, u\rangle_{\mathcal{H}} &= \int_X \left\langle C\pi\left(g^{-1}\right) v, \eta\right\rangle_{\mathcal{H}} \left\langle \eta, C\pi\left(g^{-1}\right) u\right\rangle_{\mathcal{H}} d\mu_{G/Z}(\dot{g}) \\
&= \int_X \langle v, \pi(g) C\eta\rangle_{\mathcal{H}} \langle \pi(g) C\eta, u\rangle_{\mathcal{H}} d\mu_{G/Z}(\dot{g}) \\
&= \int_X (W_{C\eta} v)(g) \overline{(W_{C\eta} u)(g)} d\mu_{G/Z}(\dot{g})
\end{aligned}$$

for all $u, v \in \operatorname{dom} C$, where $W_{C\eta} : \mathcal{H} \longrightarrow \mathcal{H}^\gamma$ is the wavelet operator associated to the vector $C\eta$. In particular,

$$Q_T(X) = W_{C\eta}^* P^\gamma(X) W_{C\eta}.$$

3 Two examples

3.1 The Heisenberg group

The Heisenberg group H is $\mathbb{R}^3$ with composition law

$$(p, q, t)(p', q', t') = \left(p + p', q + q', t + t' + \frac{pq' - qp'}{2}\right).$$

The centre of H is

$$Z = \{(0, 0, t) \mid t \in \mathbb{R}\},$$

and the quotient group G/Z is isomorphic to the Abelian group $\mathbb{R}^2$, with projection

$$q\,(p,q,t) = (p,q)\,.$$

The Heisenberg group is unimodular with Haar measure

$$\mathrm{d}\mu_{G/Z}\,(p,q) = \frac{1}{2\pi}\mathrm{d}p\mathrm{d}q.$$

Given an infinite dimensional Hilbert space $\mathcal{H}$ and an orthonormal basis $(e_n)_{n\geq 1}$, let a, a^* be the corresponding ladder operators. Define

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}}(a + a^*) \\ P &= \frac{1}{\sqrt{2}i}(a - a^*) \end{aligned}$$

It is known [2, 5] that the representation

$$\pi(p,q,t) = e^{i(t+pQ+qP)}$$

is square-integrable modulo Z and $C = 1$.

It follows from Theorem 2 that *any* π -covariant POVM Q based on $\mathbb{R}^2$ is of the form

$$Q\,(X) = \frac{1}{2\pi}\int_X e^{i(pQ+qP)}Te^{-i(pQ+qP)}\mathrm{d}p\mathrm{d}q \qquad X \in \mathcal{B}\left(\mathbb{R}^2\right)$$

for some positive trace one operator on $\mathcal{H}$. Up to our knowledge, the complete classification of the POVMs on $\mathbb{R}^2$ covariant with respect to the Heisenberg group has been an open problem till now.

3.2 The $ax + b$ group

The $ax + b$ group is the semidirect product $G = \mathbb{R} \times' \mathbb{R}_+$, where we regard $\mathbb{R}$ as additive group and $\mathbb{R}_+$ as multiplicative group. The composition law is

$$(b,a)\,(b',a') = (b + ab', aa')\,.$$

The group G is nonunimodular with left Haar measure

$$\mathrm{d}\mu_G\,(b,a) = a^{-2}\mathrm{d}b\mathrm{d}a$$

and modular function

$$\Delta\,(b,a) = \frac{1}{a}.$$

Let $\mathcal{H} = L^2((0,+\infty), \mathrm{d}x)$ and $(\pi^+, \mathcal{H})$ be the representation of G given by

$$\left[\pi^+(b,a) f\right](x) = a^{\frac{1}{2}} e^{2\pi i b x} f(ax) \qquad x \in (0,+\infty).$$

It is known [5] that π is square-integrable, and the square root of its formal degree is

$$(Cf)(x) = \Delta(0,x)^{\frac{1}{2}} f(x) = x^{-\frac{1}{2}} f(x) \qquad x \in (0,+\infty)$$

acting on its natural domain.

By means of Theorem 2 every POVM based on G and covariant with respect to π^+ is described by a positive trace one operator T according to Eq. 2. Explicitely, let $(e_i)_{i\geq 1}$ be an orthonormal basis of eigenvectors of T and $\lambda_i \geq 0$ be the corresponding eigenvalues. If $u \in L^2((0,+\infty), \mathrm{d}x)$ is such that $x^{-\frac{1}{2}} u \in L^2((0,+\infty), \mathrm{d}x)$, the π^+-covariant POVM corresponding to T is given by

$$\begin{aligned}
\langle Q_T(X) u, u \rangle_{\mathcal{H}} &= \int_X \left\langle TC\pi^+\left(g^{-1}\right) u, C\pi^+\left(g^{-1}\right) u \right\rangle_{\mathcal{H}} \mathrm{d}\mu_G(g) \\
&= \int_X \sum\nolimits_i \lambda_i \left| \left\langle C\pi^+\left(g^{-1}\right) u, e_i \right\rangle_{\mathcal{H}} \right|^2 \mathrm{d}\mu_G(g) \\
&= \sum\nolimits_i \lambda_i \int_X \left| \int_{\mathbb{R}_+} x^{-\frac{1}{2}} a^{-\frac{1}{2}} e^{-\frac{2\pi i b x}{a}} u\left(\frac{x}{a}\right) \overline{e_i(x)} \mathrm{d}x \right|^2 a^{-2} \mathrm{d}b \mathrm{d}a.
\end{aligned}$$

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