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Positive operator measures, generalised imprimitivity theorem and their applications

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Introduction

In the common textbook presentation of quantum mechanics the observables of a quantum system are represented by selfadjoint operators, or, equivalently, by spectral measures. The origin of this point of view dates back to the very beginning of quantum theory. Its rigorous mathematical formulation is mainly due to von Neumann [53], and, for a more recent and complete review, we refer to the book of Varadarajan [52].

But it is quite well known that, when particular quantum systems are considered, this approach can not give a satisfactory description of some of their physical properties. A famous example (dating back to Dirac [28]) is provided by the phase of the electromagnetic field, which is a well defined quantity in classical physics, but can not be described by any selfadjoint operator in quantum mechanics [38], [40]. This drawback in the conventional formulation of quantum theory becomes even more evident when one attempts to define a position observable for the photon. In fact, a theorem of Wightman ([56], [52]) asserts that there does not exists any selfadjoint operator describing the localisation property of the photon.

Positive operator measures were introduced just to overcome difficulties of this kind arising from the von Neumann formulation of quantum theory. Quite soon after their introduction, it became also clear [26], [38], [40], [47] that the most general description of the observables of quantum mechanics is provided by positive operator measures rather than by spectral measures. In this extended setting, the phase observable and the localisation observable for the photon are associated to positive operator measures that are *not* spectral maps, and thus can not be represented by any selfadjoint operator.

It also turned out that joint measurements of quantities which are incompatible in the von Neumann framework can be described in terms of positive operator measures (an example is provided in section 5.2).

The aim of this thesis is to give a complete characterisation of an important class of positive operator measures, namely the positive operator measures that are covariant with respect to unitary representations of a group. Indeed, in quantum physics the observables that describe a particular physical quantity are defined by means of their property of covariance with respect to a specific symmetry group. Thus, covariant positive operator measures naturally acquire a privileged role.

Since the characterisation of covariant positive operator measures requires a few deep results from abstract harmonic analysis and needs the elaboration of some specific mathematical techniques, in general we will concentrate on the mathematical point of view of the problem. Our examples and applications to quantum mechanics are not intended to give a detailed exposition of the particular physical framework to which they are addressed.

The thesis is organised as follows. Chapter 1 is of an introductory nature. In section 1.1 we give a brief account of the arising of covariant positive operator measures in quantum physics. In §1.2 we fix the mathematical set-up that will be commonly used in the subsequent chapters. In §1.3 we will sketch an application to the theory of coherent states (for more details on the topics treated in this section, we refer to [5]).

In chapters 2 and 3 we will achieve a complete characterisation of covariant positive operator measures in the following two cases:

1. if G is an abelian group, we will characterise the most general positive operator measure based on a *transitive* G-space and covariant with respect to a unitary representation of G (chapter 2);

2. if G is a generic group and Ω is a *transitive* G-space with *compact stabiliser* in G, we will classify the positive operator measures based on Ω and covariant with respect to an *irreducible* projective unitary representation of G (chapter 3).

Although they are quite specific, actually these two cases cover almost all the situations of physical interest. In particular, the first case will enable us to give a complete classification of the phase observables for a quantum electromagnetic field, while the second characterises an important class of joint observables of position and momentum, namely the covariant phase space observables.

In chapters 4 and 5 we will concentrate on positive operator measures associated to the position and momentum observables of nonrelativistic quantum mechanics. We will define such observables in terms of their properties of transformation under the action of the isochronous Galilei group. After that, we will give their characterisation and discuss their main properties. In particular, in chapter 5 we will study the problem of joint measurability of position and momentum observables. We will find under which conditions a position and a momentum observable are jointly measurable, showing that these conditions imply Heisenberg's uncertainity relation. Here we remark that in our convention position and momentum observables are covariant under the Galilei group *by definition.* Nevertheless, in many situations of physical interest one actually needs to relax the covariance requirement in the definitions of such observables. This last topic is beyond the scope of the thesis. The interested reader is referred to [11], [55] and references therein for more details.

The thesis is a re-elaboration of the results published in [13], [14], [15], [20] and [21], and in the paper [16] in preparation.

Contents

Chapter 1

Covariant positive operator measures in mathematical physics

1.1 Positive operator measures in quantum mechanics

The framework of positive operator measures naturally arises in quantum physics as the mathematical foundation of the theory of measurements. In this section, we will sketch a very brief account of the subject, with a particular emphasis on the physical meaning of positive operator measures which are covariant under symmetry transformations. For more details, we refer to [11], [26], [38], [40], [43], [47].

Quantum mechanics associates to each physical system $\mathcal{S}$ a corresponding Hilbert space $\mathcal{H}$, identifying the **states** of $\mathcal{S}$ with the elements of the set $S(\mathcal{H})$ of positive trace one operators on $\mathcal{H}$. On the other hand, if $(\Omega, \mathcal{A}(\Omega))$ is a measurable space[1], the **observables** of $\mathcal{S}$ taking values in Ω are represented by positive operator measures based on Ω and acting in $\mathcal{H}$. Here we recall that a **positive operator measure** (POM) based on Ω and acting in $\mathcal{H}$ is a map $E : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ ($\mathcal{L}(\mathcal{H})$ = the set of bounded operators on $\mathcal{H}$) such that[2]

1. $E(X)$ is a positive operator for all $X \in \mathcal{A}(\Omega)$;

[1] We recall that a measurable space is a pair $(\Omega, \mathcal{A}(\Omega))$ composed by a nonempty set Ω and a σ-algebra $\mathcal{A}(\Omega)$ of subsets of Ω.

[2] We recall that in the mathematical literature by *positive operator measure* based on Ω and acting in $\mathcal{H}$ one usually means a map $E : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ satisfying only conditions 1 and 3. If E satisfies also condition 2, then E is called to be a *positive normalised operator measure*. Our slightly imprecise abbreviated notation is justified by the fact that in the quantum theory of measurement only normalised positive operator measures are considered.

2. $E(\Omega) = I$;

3. $E\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} E(X_n)$ (in the weak sense) if $X_n \in \mathcal{A}(\Omega)$ and $X_n \cap X_m = \varnothing$ for $n \neq m$.

If $E : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ is an observable and $T \in S(\mathcal{H})$, we define

$$p_T^E(X) := \mathrm{tr}\,(TE(X)) \quad \forall X \in \mathcal{A}(\Omega). \tag{1.1}$$

By properties 1, 2 and 3 the above equation defines a probability measure p_T^E on Ω. It is interpreted as the probability distribution describing the outcomes of a measurement of E. More precisely, the number $p_T^E(X)$ is the probability that a measurement of the observable E performed on the system $\mathcal{S}$ prepared in the state T yelds a result in the subset $X \subset \Omega$.

If property 1 in the definition of POM is replaced by the stronger condition

1'. $E(X) = E(X)^* = E(X)^2$ for all $X \in \mathcal{A}(\Omega)$,

then the POM E is a **projection valued measure**, and the associated observable is said a **sharp observable**. If in addition $\Omega = \mathbb{R}$ with its Borel σ-algebra, then E is actually a spectral map. We note that in this case the mean value of the observable E measured on a state T is

$$\int \lambda \mathrm{d}p_T^E(\lambda) = \mathrm{tr}\left(T \int \lambda \mathrm{d}E(\lambda)\right) =: \mathrm{tr}\,(TA),$$

where A is the selfadjoint operator with spectral decomposition $A = \int \lambda \mathrm{d}E(\lambda)$ (here for simplicity we suppose that the domain of A is the whole space $\mathcal{H}$). We thus see that observables described by selfadjoint operators are a strict subset of the whole class of observables. For more details on the physical meaning of condition 1', we again refer to [11].

We now enrich our setting, introducing the action of a group of transformations (symmetry group), and studying how the quantum systrem $\mathcal{S}$ and its observables change under the action of the group. This will enable us to define a particular class of observables, whose properties of transformations under the action of the group are in fact the natural ones.

Thus, suppose that $E : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ is an observable, and that a transformation group G acts both on the space Ω on which E takes its values and on the quantum system $\mathcal{S}$. We denote by $S(\mathcal{H}) \ni T \longmapsto g[T] \in S(\mathcal{H})$ and by $\Omega \ni x \longmapsto g[x] \in \Omega$ the actions of an element $g \in G$ on $S(\mathcal{H})$ and Ω respectively. If G satisfies some general topological conditions, then Wigner's theorem states that G acts on the Hilbert space of the system by means of a projective unitary representation U, i.e.

$$g[T] = U(g)\,TU(g)^{-1} \quad \forall g \in G \tag{1.2}$$

for all states $T \in S(\mathcal{H})$. We now require that, when the system undergoes a transformation of G, the statistics of the measurement of E is affected by a corresponding variation. More precisely, we request that the probability distribution of the outcomes obtained measuring E satisfies

$$p^E_{g[T]}(X) = p^E_T\left(g^{-1}[X]\right) \quad \forall X \in \mathcal{A}(\Omega) \tag{1.3}$$

for all states $T \in S(\mathcal{H})$ and all transformations $g \in G$. Eqs. (1.1), (1.2) and (1.3) imply

$$E(g[X]) = U(g)\, E(X)\, U(g)^{-1} \quad \forall X \in \mathcal{A}(\Omega) \tag{1.4}$$

for all $g \in G$. Eq. (1.4) is a covariance condition imposed upon the POM E. Such a condition thus selects among all the possible observables taking values in the space Ω those which actually transform in a compatible way under the action of G.

A triple $(U, E, \mathcal{H})$ formed by a unitary representation U of G in the Hilbert space $\mathcal{H}$ and by a POM E on Ω acting in $\mathcal{H}$ which satisfies eq. (1.4) is called **system of covariance** for G based on Ω. A POM E satisfying eq. (1.4) is said to be U**-covariant**.

As a simple and clarifying example, we can consider the position observables for a quantum particle with one degree of freedom. Here the Hilbert space of the system is $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ and the outcome space is the real line $\Omega = \mathbb{R}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. We have the usual action of the group of translations $G = \mathbb{R}$ on $\mathbb{R}$ itself. We require that the statistics of the outcomes registered measuring a position observable E satisfy

$$p^E_{x[T]}(X) = p^E_T(X - x) \quad \forall x \in \mathbb{R},\ X \in \mathcal{B}(\mathbb{R}),\ T \in S(\mathcal{H}),$$

where $x[T] = e^{-ixP} T e^{ixP}$, P being the selfadjoint generator of translations. This implies

$$e^{-ixP} E(X)\, e^{ixP} = E(X + x) \quad \forall x \in \mathbb{R},\ X \in \mathcal{B}(\mathbb{R}). \tag{1.5}$$

A possible solution of the above equation is the projection valued measure Π given by

$$[\Pi(X) f](x) = \chi_X(x)\, f(x) \quad \forall f \in L^2(\mathbb{R}, \mathrm{d}x),\ X \in \mathcal{B}(\mathbb{R}).$$

We note that Π is the spectral map associated to the selfadjoint generator Q of momentum boosts, with $(Qf)(x) = xf(x)$. Neverthless, we shall see in the next chapter that there are many other solutions of eq. (1.5), for which the condition 1' does not hold in general. By the way, we note that the covariance under translations is not sufficient to define a position observable; also invariance under boosts is needed. We shall explore this with more details in chapter 4.

In the following, we will be mainly concerned with the solution of eq. (1.4), i.e. with the classification of the systems of covariance for a group G based on a G-space Ω. We will always assume that the action of G on Ω is *transitive.* Under this essential hypothesis (and some general topological assumptions), we will find the most general solution in the following two cases:

1. G is abelian and U is an arbitrary unitary representation of G (chapter 2);

2. G is generic, U is an irreducible projective unitary representation of G, and the stabilizer of Ω in G is compact (chapter 3).

The first case covers, for example, the class of position observables just described, while the second characterises an important class of joint observables of position and momentum, namely the covariant phase space observables (see §3.5).

The next section lays down the mathematical set-up which will be the starting point for our solution of this classification problem.

1.2 The induced representation and the generalised imprimitivity theorem

In the following, we will always be concerned with topological spaces endowed with their Borel σ-algebra. If Ω is a topological space, we denote by $\mathcal{B}(\Omega)$ the σ-algebra of its Borel subsets.

We will always assume that G is a Hausdorff locally compact second countable topological group acting continuously and transitively on a Hausdorff locally compact space Ω (transitive G-space). We denote by e the identity element of G. Fixed a point $x \in \Omega$, Ω is canonically identified with the quotient space G/H_x, $H_x \subset G$ being the stability subgroup at x. By virtue of this identification, for the rest of this section we shall assume that $\Omega = G/H$, with H a closed subgroup in G. The canonical projection $G \longrightarrow G/H$ is denoted by π; the equivalence class of an element $g \in G$ in G/H is denoted alternatively by $\pi(g)$ or $\dot{g}$. The action of an element $a \in G$ on a point $\dot{g} \in G/H$ is clearly $a[\dot{g}] = \pi(ag)$.

We assume that G/H *admits an invariant measure* $\mu_{G/H}$. We recall that this is equivalent to assume that the modular function Δ_G of G restricts on H to the modular function Δ_H of H. If the invariant measure $\mu_{G/H}$ exists, then it is unique up to a constant. We let μ_G be a fixed left Haar measure of G. The following fact will often be used: a set $X \in \mathcal{B}(G/H)$ is $\mu_{G/H}$-negligible if and only if $\pi^{-1}(X)$ is μ_G-negligible. For more details on these facts, we refer to [33], [34].

By a **unitary representation** (or simply a **representation**) we mean a weakly continuous unitary representation acting in a separable Hilbert space. If $\mathcal{H}$ is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ its scalar product, linear in the second argument; when clear from the context, we drop the subscript $\mathcal{H}$.

Let rep (H) and rep (G) be the set of the unitary representations of H and G respectively. We now describe a canonical construction which will allow us to associate:

- to each unitary representation σ of H a unitary representation of G, called the **representation of G induced by σ** and denoted by $\mathrm{ind}_H^G(\sigma)$;
- to each map A intertwining[3] σ with σ' a map $\tilde{A}$ intertwining $\mathrm{ind}_H^G(\sigma)$ with $\mathrm{ind}_H^G(\sigma')$ in such a way that: a) $\widetilde{AB} = \tilde{A}\tilde{B}$ if A intertwines σ with σ' and B intertwines σ' with σ''; b) $\widetilde{A^*} = \tilde{A}^*$.

Suppose σ is a representation of H in the Hilbert space $\mathcal{K}$. Let $\mathcal{H}^\sigma$ be the space of functions $f : G \longrightarrow \mathcal{K}$ such that

1. f is weakly μ_G-measurable;
2. for all $h \in H$ and $g \in G$
$$f\left(gh\right) = \sigma\left(h\right)^{-1} f\left(g\right);$$
3. $$\int_{G/H} \left\| f\left(g\right)\right\|^2 \mathrm{d}\mu_{G/H}\left(\dot{g}\right) < \infty.$$

We identify functions in $\mathcal{H}^\sigma$ which are equal μ_G-a.e.[4]. Endowed with the scalar product

$$\langle f_1, f_2\rangle_{\mathcal{H}^\sigma} = \int_{G/H} \langle f_1\left(g\right), f_2\left(g\right)\rangle \, \mathrm{d}\mu_{G/H}\left(\dot{g}\right)$$

$\mathcal{H}^\sigma$ becomes a separable Hilbert space (see [27]). There is a natural unitary representation of G in $\mathcal{H}^\sigma$, in which G acts on $\mathcal{H}^\sigma$ by left translations. We denote it by λ^σ:

$$\left[\lambda^\sigma\left(a\right) f\right]\left(g\right) = f\left(a^{-1}g\right) \quad \text{for } \mu_G\text{-a.a. } g \in G.$$

[3]We recall that, if σ and σ' are representations of H acting in the Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, a bounded operator $A : \mathcal{K} \longrightarrow \mathcal{K}'$ **intertwines** σ and σ' if

$$A\sigma(h) = \sigma'(h)A \qquad \forall h \in H.$$

[4]Here and in the following 'a.e.' is the acronym of 'almost everywhere'. Likewise, 'a.a.' is the abbreviated form of 'almost all'.

for all $a \in G$.

We then define $\mathrm{ind}_H^G(\sigma) = \lambda^\sigma$. If A intertwines σ with σ', we let $\tilde{A} : \mathcal{H}^\sigma \longrightarrow \mathcal{H}^{\sigma'}$ be defined by

$$\left(\tilde{A}f\right)(g) = Af(g).$$

It is immediately checked that $\tilde{A}$ satisfies the required properties.

Remark 1.2.1 *Note that if $H = \{1\}$ and σ is the trivial one-dimensional representation of H, then $\mathrm{ind}_H^G(\sigma)$ is the left regular representation λ of G. We recall that this representation acts by left translations in the Hilbert space $L^2(G, \mu_G)$:*

$$[\lambda(a) f](g) = f\left(a^{-1}g\right) \quad \text{for } \mu_G\text{-a.a. } g \in G.$$

for all $f \in L^2(G, \mu_G)$, $a \in G$. In the following, we will often drop the adjective 'left' in referring to this representation.

An equivalent realisation of ind_H^G is now given. Fix a Borel section $s : G/H \longrightarrow G$. Define

$$(Vf)(x) = f(s(x)) \quad \text{for } \mu_{G/H}\text{-a.a. } x \in G/H$$

for all $f \in \mathcal{H}^\sigma$. Then, $Vf \in L^2\left(G/H, \mu_{G/H}; \mathcal{K}\right)$, and the linear operator $V : \mathcal{H}^\sigma \longrightarrow L^2\left(G/H, \mu_{G/H}; \mathcal{K}\right)$ is unitary. By means of V, we transfer λ^σ to a unitary representation U^σ of G in $L^2\left(G/H, \mu_{G/H}; \mathcal{K}\right)$. Its action on $\phi \in L^2\left(G/H, \mu_{G/H}; \mathcal{K}\right)$ is

$$[U^\sigma(a)\phi](x) = \sigma\left(s(x)^{-1} as\left(a^{-1}[x]\right)\right)\phi\left(a^{-1}[x]\right)$$

for all $a \in G$.

In the following, we will mainly use the realisation of $\mathrm{ind}_H^G(\sigma)$ in $\mathcal{H}^\sigma$, although sometimes we will refer also to the second construction.

In $\mathcal{H}^\sigma$ we define the projection valued measure P^σ based on G/H which maps each $X \in \mathcal{B}(G/H)$ into the multiplication by the corresponding characteristic function χ_X:

$$[P^\sigma(X) f](g) = \chi_X(\dot{g}) f(g) \quad \forall f \in \mathcal{H}^\sigma. \tag{1.6}$$

Clearly,

$$\lambda^\sigma(g) P^\sigma(X) \lambda^\sigma(g)^{-1} = P^\sigma(g[X]) \quad \forall X \in \mathcal{B}(G/H), \forall g \in G.$$

A triple $(U, P, \mathcal{H})$, in which U is a unitary representation of G in the Hilbert space $\mathcal{H}$, $P : \mathcal{B}(G/H) \longrightarrow \mathcal{L}(\mathcal{H})$ is a projection valued measure, and

$$U(g) P(X) U(g)^{-1} = P(g[X]) \quad \forall X \in \mathcal{B}(G/H), \forall g \in G,$$

is called **system of imprimitivity** for G based on G/H. The triple $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ defined above is the **canonical system of imprimitivity** induced by the representation σ of H.

We can now state the fundamental theorem on which our classification of covariant POM's will be based. It is a generalisation of the imprimitivity theorem of Mackey ([48], [33]), and, in its first proof given by Cattaneo in [24], it is a consequence of Mackey's theorem and of the dilation theorem of Naimark. A direct proof, which includes the proof of the Mackey imprimitivity theorem, is given in [19]. We recall from the previous section that a system of covariance for G based on G/H is a triple $(U, E, \mathcal{H})$ made up by a unitary representation U of G in the Hilbert space $\mathcal{H}$ and a POM E on G/H acting in $\mathcal{H}$ and satisfying the covariance condition of eq. (1.4).

Theorem 1.2.2 (Generalised imprimitivity theorem) *Let $(U, E, \mathcal{H})$ be a system of covariance for G based on G/H. There exists a representation σ of H such that*

1. *there is an isometry $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$, with $WU(g) = \lambda^\sigma(g) W \ \forall g \in G$, such that*
$$E(X) = W^* P^\sigma(X) W \quad \forall X \in \mathcal{B}(G/H);$$
2. *the linear hull of the set*
$$\{P^\sigma(X) Wv \mid v \in \mathcal{H},\ X \in \mathcal{B}(G/H)\}$$
is dense in $\mathcal{H}^\sigma$.

Moreover, the representation σ satisfying 1 and 2 is unique up to equivalence.

Conversely, given a representation U of G on $\mathcal{H}$, suppose that for some $\sigma \in \mathrm{rep}\,(H)$ there exists an isometry $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$ intertwining U with λ^σ. Then, it is immediately checked that

$$E(X) := W^* P^\sigma(X) W \quad \forall X \in \mathcal{B}(G/H)$$

defines a system of covariance $(U, E, \mathcal{H})$.

By virtue of the above theorem, the problem of classifying all the systems of covariance for G based on G/H reduces to characterising the representations of G which are contained in representations induced from H. This is directly related to the problem of diagonalising the induced representation, hence in particular to Plancherel theory. Thus, as we shall see, a general solution is achievable only for particular choices of H, such that a canonical method for diagonalising $\mathrm{ind}_H^G(\sigma)$ $(\sigma \in \mathrm{rep}\,(H))$ can be worked out.

Remark 1.2.3 *Suppose $(U, E, \mathcal{H})$ and $(U', E', \mathcal{H}')$ are two systems of covariance for G based on G/H. Then they are **equivalent** if there exists a unitary operator $V : \mathcal{H} \longrightarrow \mathcal{H}'$ such that $U'(g) = VU(g)V^{-1}\ \forall g \in G$ and $E'(X) = VE(X)V^{-1}\ \forall X \in \mathcal{B}(G/H)$.*

Remark 1.2.4 *Suppose* Ω *is a locally compact second countable Hausdorff space, and let* $C_c(\Omega)$ *be the linear space of continuous complex valued functions on* Ω *with compact support. If* $E : \mathcal{B}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ *is a POM, for* $\varphi \in C_c(\Omega)$ *we define the following bounded operator on* $\mathcal{H}$

$$E(\varphi) := \int_\Omega \varphi(x)\, dE(x),$$

where the integral is understood in the weak sense. It is known that the map $\varphi \longmapsto E(\varphi)$ *defines uniquely the POM* E*. Indeed, if* $u, v \in \mathcal{H}$*, let* $\mu_{u,v}$ *be the complex measure given by*

$$\mu_{u,v}(X) = \langle u, E(X)\, v\rangle \quad \forall X \in \mathcal{B}(\Omega). \tag{1.7}$$

By the Riesz representation theorem, the complex measure $\mu_{u,v}$ *can be recovered from the bounded linear functional it induces on* $C_c(\Omega)$*, i.e. from the mapping*

$$C_c(\Omega) \ni \varphi \longmapsto \int_\Omega \varphi(x)\, d\mu_{u,v}(x) \equiv \langle u, E(\varphi)\, v\rangle$$

(here we used the definition of $E(\varphi)$*). Hence, the mapping* $\varphi \longmapsto E(\varphi)$ *determines all the complex measures* $\mu_{u,v}$ $(u, v \in \mathcal{H})$*, and these in turn determine the POM* E *by means of (1.7). For more details, we refer to [7]. In the following, we will often use this alternative description of* E*.*

If $\Omega = G/H$*, denoting by* $\varphi^g(x) := \varphi\left(g^{-1}[x]\right)$ *the action of an element* $g \in G$ *on a function* $\varphi \in C_c(G/H)$*, covariance condition (1.4) is equivalent to the following*

$$E(\varphi^g) = U(g)\, E(\varphi)\, U(g)^{-1} \quad \forall \varphi \in C_c(G/H),\ g \in G.$$

If P^σ *is the projection valued measure of eq. (1.6), we have, for* $\varphi \in C_c(G/H)$*,*

$$[P^\sigma(\varphi)\, f](g) = \varphi(\dot{g})\, f(g) \quad \forall f \in \mathcal{H}^\sigma.$$

Remark 1.2.5 *Let* σ *be a representation of* H *in the Hilbert space* $\mathcal{K}$*. For* $\varphi \in C_c(G)$ *and* $v \in \mathcal{K}$*, let*

$$f_{\varphi v}(g) := \int_H \varphi(gh)\, \sigma(h)\, v\, d\mu_H(h) \qquad \forall g \in G$$

(here μ_H *is a Haar measure of* H*). It is easily checked that* $f_{\varphi v}$ *is a continuous function in* $\mathcal{H}^\sigma$*, whose support is compact modulo* H*. Moreover, if* $\mathcal{D} \subset \mathcal{K}$ *is a subset which is total in* $\mathcal{K}$*, it is known that the set*

$$\{f_{\varphi v} \mid \varphi \in C_c(G),\ v \in \mathcal{D}\}$$

is total in $\mathcal{H}^\sigma$ *(see [33]).*

Remark 1.2.6 *In chapter 3, we will deal also with* ***projective unitary representations*** *of the group G (sometimes abbreviated in 'projective representations'). We recall that such a representation acting in a Hilbert space $\mathcal{H}$ is a weakly measurable map $G \ni g \longmapsto U(g) \in \mathcal{U}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ being the unitary group of $\mathcal{H}$, such that $U(e) = 1$ and $U(g_1 g_2) = m(g_1, g_2) U(g_1) U(g_2)$. The measurable map $m : G \times G \longrightarrow \mathbb{T}$, $\mathbb{T}$ being the set of complex numbers with modulus one, is the* ***multiplier*** *of U (see [52] for more details; in particular, if the multiplier m is trivial, there exists a measurable map $a : G \longrightarrow \mathbb{T}$ such that $G \longmapsto a(g)U(g)$ is a strongly continuous unitary representation of G). As in the unitary case, a POM $E : \mathcal{B}(G/H) \longrightarrow \mathcal{H}$ is U-covariant if $U(g) E(X) U(g)^{-1} = E(g[X])$ for all $X \in \mathcal{B}(G/H)$ and $g \in G$.*

1.3 Systems of covariance and coherent states

Let G, H and G/H be as in the previous section. In addition, suppose $U \in \mathrm{rep}\,(G)$ is fixed. Let $\mathcal{H}$ be the Hilbert space of U. We are now interested in the U-covariant POM's that are expressible by means of an operator density with respect to some measure on G/H. In other words, we are looking for those U-covariant POM's E based on G/H such that there exist a positive Borel measure ν on G/H and a weakly ν-measurable map $G/H \ni x \longmapsto E(x) \in \mathcal{L}(\mathcal{H})$ satisfying

$$E(X) = \int_X E(x)\, \mathrm{d}\nu(x) \quad \forall X \in \mathcal{B}(G/H). \tag{1.8}$$

Here and in the following, our operator valued integrals are always understood in the weak sense. If eq. (1.8) holds, then we have the following resolution of the identity

$$I = \int_{G/H} E(x)\, \mathrm{d}\nu(x), \tag{1.9}$$

so that $E(x)$, $x \in G/H$, define a family of **generalised coherent states** in the sense of [5]. This terminology is justified by the following fact: if there exists a measurable map $G/H \ni x \longmapsto \psi(x) \in \mathcal{H}$, with $\|\psi(x)\| = 1$ a.e., such that $E(x)$ is the orthogonal projection along $\psi(x)$, then eq. (1.9) reads[5]

$$I = \int_{G/H} \langle \psi(x), \cdot \rangle\, \psi(x)\, \mathrm{d}\nu(x),$$

[5] We recall that our inner product is linear in the second argument. Eq. (1.9) is also equivalent to

$$\langle u, v \rangle = \int \langle u, \psi(x) \rangle \langle \psi(x), v \rangle\, \mathrm{d}\nu(x) \quad \forall u, v \in \mathcal{H}.$$

thus showing the connection with classical coherent states [26].

Theorem 1.3.2 below is a consequence of the result found by Cattaneo in [25]. Here we give a different and simplified proof, which, up to our knowledge, is new. It is based on the following lemma.

Lemma 1.3.1 *If eq. (1.8) holds for the U-covariant POM E, then there exists a weakly continuous map $G/H \ni x \longmapsto \widetilde{E}(x) \in \mathcal{L}(\mathcal{H})$ such that*

1. $$U(g)\,\widetilde{E}(x)\,U(g)^{-1} = \widetilde{E}(g[x]) \quad \forall g \in G, \forall x \in G/H;$$

2. $$E(X) = \int_X \widetilde{E}(x)\, d\mu_{G/H}(x) \quad \forall X \in \mathcal{B}(G/H).$$

Proof. We first show that in eq. (1.8) $E(x)$ is a positive operator for ν-a.a. x, and ν can always be chosen to be the invariant measure $\mu_{G/H}$. First of all, possibly redefining ν, we can assume $E(x) \neq 0$ for ν-a.a. x. Let $(v_n)_{n\geq 1}$ be a countable sequence of vectors which is dense in $\mathcal{H}$. Since, for all $n \geq 1$, $\langle E(X)\, v_n, v_n\rangle \geq 0$ for all $X \in \mathcal{B}(G/H)$, it follows from eq. (1.8) that there is a ν-null set $N \in \mathcal{B}(G/H)$ such that $\langle E(x)\, v_n, v_n\rangle \geq 0$ for all $n \geq 1$ and $x \notin N$. By continuity, $E(x) \geq 0$ for all $x \notin N$. For each n, we define the bounded measures

$$\mu_n(X) = \|v_n\|^{-2} \langle v_n, E(X)\, v_n\rangle \quad \forall X \in \mathcal{B}(G/H)$$

and

$$\mu = \sum_n 2^{-n} \mu_n.$$

We then have the equivalence

$$\mu(X) = 0 \Longleftrightarrow E(X) = 0,$$

and, since $E(g[X]) = U(g)\, E(X)\, U(g)^{-1}$,

$$\mu(X) = 0 \Longleftrightarrow \mu(g[X]) = 0.$$

It follows that μ is equivalent to the invariant measure $\mu_{G/H}$ (see [33]). On the other hand, by eq. (1.8) and monotone convergence theorem, μ has density

$$x \longmapsto \sigma(x) := \sum_n 2^{-n} \|v_n\|^{-2} \langle v_n, E(x)\, v_n\rangle$$

with respect to ν, and, since $\sigma(x) > 0$ for ν-a.a. x, μ and ν are equivalent. Let ρ be the density of ν with respect to $\mu_{G/H}$, and define $\widetilde{E}(x) = \rho(x)\, E(x)$. We then have

$$E(X) = \int_X \widetilde{E}(x)\, \mathrm{d}\mu_{G/H}(x) \quad \forall X \in \mathcal{B}(G/H)$$

as claimed.

The covariance condition on E easily implies that for all $g \in G$ there is a $\mu_{G/H}$-null set N_g such that

$$U(g)\,\widetilde{E}(x)\,U(g)^{-1} = \widetilde{E}(g[x]) \quad \forall x \notin N_g. \tag{1.10}$$

We now show that N_g can be chosen independent on $g \in G$. This will complete the proof of the lemma. Note that, since

$$\exp\left[\pm i\widetilde{E}(x)\right] = \sum_{k=0}^{\infty} \frac{(\pm i)^k}{k!}\widetilde{E}(x)^k,$$

the convergence being in the uniform norm of $\mathcal{L}(\mathcal{H})$, the maps $x \longmapsto \exp\left(\pm i\widetilde{E}(x)\right)$ are weakly $\mu_{G/H}$-measurable. In fact, it suffices to show that each map $x \longmapsto \widetilde{E}(x)^k$ is weakly $\mu_{G/H}$-measurable, and this is seen by induction on k: fixed an orthonormal basis $(\phi_n)_{n\geq 1}$ of $\mathcal{H}$, we have

$$\left\langle u, \widetilde{E}(x)^k v\right\rangle = \sum_n \left\langle u, \widetilde{E}(x)\,\phi_n\right\rangle \phi_n, \left\langle \widetilde{E}(x)^{k-1} v\right\rangle \quad \forall x \in G/H$$

and by the inductive hypothesis the right hand side is the sum of $\mu_{G/H}$-measurable functions, hence is $\mu_{G/H}$-measurable. Inserting again $I = \sum_n \langle \cdot, \phi_n\rangle\,\phi_n$ between the composed operators, we see that the maps

$$\begin{aligned} g &\longmapsto 2I + U(g)^{-1}\exp\left[i\widetilde{E}(\dot{g})\right]U(g) + \text{h.c.} =: B^{+}(g) \\ g &\longmapsto 2I + iU(g)^{-1}\exp\left[i\widetilde{E}(\dot{g})\right]U(g) + \text{h.c.} =: B^{-}(g) \end{aligned}$$

are weakly μ_G-measurable. Moreover, $B^{\pm}(g) \geq 0$ and $\|B^{\pm}(g)\| \leq 4$ for all $g \in G$. For $n \geq 1$, we can thus define the positive Borel measures $\alpha_n^{\pm}$ on G, given by

$$\int_G f(g)\,\mathrm{d}\alpha_n^{\pm}(g) = \int_G f(g)\left\langle v_n, B^{\pm}(g)\,v_n\right\rangle \mathrm{d}\mu_G(g) \quad \forall f \in C_c(G).$$

For all $a \in G$, by eq. (1.10) we have

$$B^{\pm}(ag) = B^{\pm}(g) \quad \text{for a.a. } g \in G,$$

and so $\alpha_n^{\pm}$ are invariant measures on G. This implies $\alpha_n^{\pm} = c_n^{\pm}\mu_G$ for some constants $c_n^{\pm}$, hence $\langle B^{\pm}(g)\,v_n, v_n\rangle$ is a constant for a.a. g. It follows that there is a μ_G-null set Z and operators $C^{\pm}$ such that

$$B^{\pm}(g) = C^{\pm} \quad \forall g \notin Z.$$

Since

$$U(g)^{-1}\exp\left[i\widetilde{E}(\dot{g})\right]U(g) = \frac{1}{2i}\left(C^{+} - iC^{-}\right) + (i-1)\,I \quad \text{for a.a. } g \in G,$$

by spectral theorem $g \longmapsto U(g)^{-1} \widetilde{E}(\dot{g}) U(g)$ is constant almost everywhere. Our claim is thus proved. ■

The next theorem characterises those U-covariant POM's which define a family of generalised coherent states.

Theorem 1.3.2 *Suppose E is a U-covariant POM based on G/H. Let σ and W be as in Theorem 1.2.2. Then, E admits the representation of eq. (1.8) if and only there exists a bounded operator $A : \mathcal{H} \longrightarrow \mathcal{K}$, $\mathcal{K}$ being the Hilbert space of σ, such that*

1. $$AU(h) = \sigma(h) A \quad \forall h \in H; \tag{1.11}$$

2. $$(Wv)(g) = AU(g)^{-1} v \quad \text{for a.a. } g \in G. \tag{1.12}$$

Proof. If A satisfies the conditions in the statement, then

$$\begin{aligned} \langle u, E(X) v \rangle_{\mathcal{H}} &= \langle Wu, P^{\sigma}(X) Wv \rangle_{\mathcal{H}^{\sigma}} \\ &= \int_X \left\langle AU(g)^{-1} u, AU(g)^{-1} v \right\rangle_{\mathcal{K}} \mathrm{d}\mu_{G/H}(\dot{g}) \end{aligned}$$

and eq. (1.8) follows with

$$E(\dot{g}) := U(g) A^* A U(g)^{-1} \quad \forall g \in G.$$

Conversely, if eq. (1.8) holds, then by the previous lemma we can take $\nu = \mu_{G/H}$ and $E(\dot{g}) = U(g) T U(g)^{-1}$ for some fixed positive operator T, where T commutes with the representation σ' of H obtained restricting U to H. For all $v \in \mathcal{H}$, define

$$\left(W'v\right)(g) = T^{1/2} U(g)^{-1} v \quad \forall g \in G.$$

By eq. (1.9), W' is an isometry $\mathcal{H} \longrightarrow \mathcal{H}^{\sigma'}$. It clearly intertwines U with $\lambda^{\sigma'}$. Let P be the orthogonal projection onto the closed subspace $\overline{\mathrm{span}}\{P^{\sigma'}(X) W'v \mid v \in \mathcal{H},\, X \in \mathcal{B}(G/H)\}$. Then, as P commutes with $\lambda^{\sigma'}$ and $P^{\sigma'}$, the imprimitivity theorem of Mackey implies that there exists an orthogonal projection $P_{\mathcal{H}}$ of $\mathcal{H}$ commuting with σ' and such that

$$(Pf)(g) = P_{\mathcal{H}} f(g) \quad \forall f \in \mathcal{H}^{\sigma'}.$$

By the uniqueness statement in Theorem 1.2.2, we can assume $\sigma = \sigma'|_{P_{\mathcal{H}}\mathcal{H}}$, the restriction of σ' to the subspace $\mathcal{K} = P_{\mathcal{H}}\mathcal{H}$, and $W = W'$. Since $PW' = W'$, we have $P_{\mathcal{H}} T^{1/2} v = (PW'v)(e) = (W'v)(e) = T^{1/2} v$ for all $v \in \mathcal{H}$. Thus, $\mathrm{ran}\, T^{1/2} \subset \mathcal{K}$. If we set $A = T^{1/2}$, then A satisfies eqs. (1.11) and (1.12), and the proof of the theorem is complete. ■

An operator $A : \mathcal{H} \longrightarrow \mathcal{K}$ as in the statement of the theorem above is called **generalised wavelet operator** (see [5]). Note that an intertwining isometry $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$ is expressible in the form of eq. (1.12), with $A : \mathcal{H} \longrightarrow \mathcal{K}$ bounded and satisfying eq. (1.11), if and only if the subspace $W\mathcal{H} \subset \mathcal{H}^\sigma$ is a reproducing kernel Hilbert space of continuous functions on G (recall that a reproducing kernel Hilbert space on G is a Hilbert space of functions in which the evaluation maps at each point $g \in G$ are continuous functionals). We thus find in a different way the result found by Cattaneo in ([25]).

Chapter 2

The abelian case

2.1 Introduction

Throughout all this chapter, G will be a Hausdorff locally compact *abelian* group satisfying the second countability axiom, and H will be a closed subgroup in G.

In the following, we describe all the systems of covariance for G based on G/H. First of all, given $U \in \mathrm{rep}\,(G)$, we will find a necessary and sufficient condition in order that U admits covariant POM's based on G/H. To give a brief explanation of this point, let $\widehat{G}$ be the group of unitary characters of G, and $[M(\widehat{G})]$ be the partially ordered set of equivalence classes of positive Borel measures on $\widehat{G}$ (here $[\rho] \leq [\nu]$ iff ρ has density with respect to ν). As we shall explain with more details in section 2.4, the Stone-Naimark-Ambrose-Godement (SNAG) theorem canonically associates to U a unique measure class $\mathcal{C}_U \in [M(\widehat{G})]$. We will construct a canonical order preserving map $\Phi_H : [M(\widehat{G})] \longrightarrow [M(\widehat{G})]$ such that the following holds: U admits covariant POM's based on G/H if and only if $[\mathcal{C}_U] \leq \Phi_H([\mathcal{C}_U])$. If this condition is satisfied, the next step consists in characterising the most general U-covariant POM based on G/H. We will see that such a POM is described in terms of a family $W_x : E_x \to E$ of isometries, where the index x runs over the dual group $\widehat{G}$, $\dim E_x$ equals the multiplicity of the character x in U and E is a fixed (infinite dimensional) Hilbert space. Since our classification of covariant POM's is based on the generalised imprimitivity theorem, an essential point is the definition of a unitary transform Σ which diagonalises $\mathrm{ind}_H^G(\sigma)$ for $\sigma \in \mathrm{rep}\,(H)$. This sort of "generalised Fourier transform" is the subject of §2.3. In §2.4 we will state the basic results of this chapter. Finally, as an application in §2.5 we give three examples:

1. the *regular* representation of the real line, where the positive operator measures describe a class of *position observables* in one dimension (§2.5.1);

2. the *number*-representation of the torus, where the positive operator measures describe the *phase observables* for a single mode bosonic field (§2.5.2);

3. the tensor product of two *number*-representations of the torus, where the positive operator valued measures describe the *phase difference observables* (§2.5.3).

In the literature, the problem of classifying the U-covariant POM's for the abelian group G had been solved by Holevo in the special case $H = \{e\}$ [41]. Our results are then a generalisation of the results of Holevo. In addition, in the case $H = \{e\}$ our approach gives a description of covariant POM's which is more manageable than the already known one.

The material in this chapter is taken from [21].

2.2 Notations

All the groups considered in this chapter are Hausdorff, locally compact, second countable and abelian. If Ω is a locally compact second countable Hausdorff space, by a *measure* on Ω we always mean a positive measure defined on the Borel σ-algebra $\mathcal{B}(\Omega)$ of Ω which is finite on compact sets. If $\mathcal{H}$ is a Hilbert space, we denote by $C_c(\Omega;\mathcal{H})$ the space of functions $\varphi : \Omega \longrightarrow \mathcal{H}$ which are continuous and with compact support. If $\mathcal{H} = \mathbb{C}$, we use the standard abbreviated notation $C_c(\Omega)$ for $C_c(\Omega;\mathbb{C})$.

In the sequel we shall use rather freely basic results of harmonic analysis on abelian groups, as exposed, for example, in refs. [27] and [33].

We fix a group G and a closed subgroup H. We denote by $\widehat{G}$ and $\widehat{H}$ the corresponding dual groups and by $\langle x, g\rangle$ the canonical pairing.

We recall that

$$\pi : G \longrightarrow G/H, \qquad \pi(g) = \dot{g}$$

is the canonical projection onto the quotient group G/H. If $a \in G$ and $\dot{g} \in G/H$, the action of a on the point $\dot{g}$ is $a[\dot{g}] = \dot{a}\dot{g}$.

Let $H^{\perp}$ be the annihilator of H in $\widehat{G}$, that is

$$H^{\perp} = \left\{ y \in \widehat{G} \mid \langle y, h\rangle = 1 \ \forall h \in H \right\}.$$

The group $H^{\perp}$ is a closed subgroup of $\widehat{G}$ and $\widehat{G/H}$ can be identified (and we will do) with $H^{\perp}$ by means of

$$\langle y, \dot{g}\rangle := \langle y, g\rangle \quad \forall y \in H^{\perp}, \forall \dot{g} \in G/H.$$

Since $H^{\perp}$ is closed, we can consider the quotient group $\widehat{G}/H^{\perp}$. We denote by

$$q : \widehat{G} \longrightarrow \widehat{G}/H^{\perp}, \qquad q(x) = \dot{x}$$

the canonical projection. The group $\widehat{H}$ can be identified (and we will do) with the quotient group $\widehat{G}/H^{\perp}$ by means of

$$\langle \dot{x}, h \rangle := \langle x, h \rangle \quad \forall \dot{x} \in \widehat{G}/H^{\perp}, \forall h \in H.$$

Let μ_G, μ_H and $\mu_{G/H}$ be fixed Haar measures on G, H and G/H, respectively.

We denote by $\mu_{H^{\perp}}$ the Haar measure on $H^{\perp}$ such that the Fourier-Plancherel cotransform $\overline{\mathcal{F}}_{G/H}$ is a unitary operator from $L^2\left(G/H, \mu_{G/H}\right)$ onto $L^2\left(H^{\perp}, \mu_{H^{\perp}}\right)$, where $\overline{\mathcal{F}}_{G/H}$ is given by

$$\left(\overline{\mathcal{F}}_{G/H} f\right)(y) = \int_{G/H} \langle y, \dot{g} \rangle f(\dot{g}) \, \mathrm{d}\mu_{G/H}(\dot{g}) \quad \text{for a.a. } y \in H^{\perp}$$

for all $f \in \left(L^1 \cap L^2\right)\left(G/H, \mu_{G/H}\right)$.

Given $\varphi \in C_c\left(\widehat{G}\right)$, let

$$\widetilde{\varphi}(\dot{x}) := \int_{H^{\perp}} \varphi(xy) \, \mathrm{d}\mu_{H^{\perp}}(y) \quad \forall \dot{x} \in \widehat{G}/H^{\perp}.$$

It is well known that $\widetilde{\varphi}$ is in $C_c\left(\widehat{G}/H^{\perp}\right)$ and that $\widetilde{\varphi} \geq 0$ if $\varphi \geq 0$. Given a measure ν on $\widehat{G}/H^{\perp}$, the map

$$C_c\left(\widehat{G}\right) \ni \varphi \longmapsto \int_{\widehat{G}/H^{\perp}} \widetilde{\varphi}(\dot{x}) \, \mathrm{d}\nu(\dot{x}) \in \mathbb{C} \tag{2.1}$$

is linear and positive. Hence, by Riesz-Markov theorem, there is a unique measure $\widetilde{\nu}$ on $\widehat{G}$ such that

$$\int_{\widehat{G}} \phi(x) \, \mathrm{d}\widetilde{\nu}(x) = \int_{\widehat{G}/H^{\perp}} \mathrm{d}\nu(\dot{x}) \int_{H^{\perp}} \phi(xy) \, \mathrm{d}\mu_{H^{\perp}}(y)$$

for all $\phi \in L^1\left(\widehat{G}, \widetilde{\nu}\right)$. One can check that the correspondence $\nu \longmapsto \widetilde{\nu}$ preserves equivalence and orthogonality of measures.

Given a finite measure μ on $\widehat{G}$, we denote by μ^q the image measure of μ with respect to q, i.e. the measure on $\widehat{G}/H^{\perp}$ given by

$$\mu^q(A) = \mu\left(q^{-1}(A)\right) \quad \forall A \in \mathcal{B}\left(\widehat{G}/H^{\perp}\right).$$

The following well known theorem of harmonic analisys characterises the most general unitary representation of the abelian group G. It is stated in this form for example in [33, Theorem 7.40] (see also [27, Theorem 22.15.1]). We recall that if ρ is a measure on $\widehat{G}$ and F is a Hilbert space, the **diagonal representation** $U^{\rho,F}$ of G in the space $L^2\left(\widehat{G}, \rho; F\right)$ is defined by

$$\left(U^{\rho,F}\phi\right)(x) = \langle x, g \rangle \phi(x) \quad \text{for } \rho\text{-a.a } x \in \widehat{G}$$

for all $\phi \in L^2\left(\widehat{G}, \rho; F\right)$ and $g \in G$.

Theorem 2.2.1 (Stone, Naimark, Ambrose, Godement) *Let U be a unitary representation of the abelian group G. Then there exists a family of disjoint measures $(\rho_j)_{j\in\mathbb{Z}_+\cup\{\infty\}}$ on $\widehat{G}$ and a family of Hilbert spaces $(F_k)_{j\in\mathbb{Z}_+\cup\{\infty\}}$ with $\dim F_k = k$ such that U is unitarily equivalent to the direct sum $\bigoplus_{j\in\mathbb{Z}_+\cup\{\infty\}} U^{\rho_j,F_j}$.*

If $\left(\rho'_j\right)_{j\in\mathbb{Z}_+\cup\{\infty\}}$ is another family of disjoint measures on $\widehat{G}$ such that U is unitarily equivalent to the direct sum $\bigoplus_{j\in\mathbb{Z}_+\cup\{\infty\}} U^{\rho'_j,F'_j}$, then ρ_j and ρ'_j are equivalent measures for all $j \in \mathbb{Z}_+ \cup \{\infty\}$.

We say that $U \in \mathrm{rep}\,(G)$ has **uniform multiplicity** $m \in \mathbb{Z}_+ \cup \{\infty\}$ if, with the notations of the above theorem, we have $\rho = 0$ for all $j \neq m$.

We now fix a representation U of G acting on a Hilbert space $\mathcal{H}$. Our aim is to describe all the positive operator measures covariant with respect to U. From the generalised imprimitivity theorem the following fact immediately follows.

Theorem 2.2.2 *A POM E based on G/H and acting on $\mathcal{H}$ is covariant with respect to U if and only if there exists a representation σ of H and an isometry W intertwining U with $\mathrm{ind}_H^G(\sigma)$ such that*

$$E(\omega) = W^* P^\sigma(\omega) W$$

for all $\omega \in C_c(G/H)$.

Note that here and in the rest of this chapter we define the POM E by means of its representation as a linear form on $C_c(G/H)$, as explained in Remark 1.2.4. This is done in order to avoid some technical difficulties (like problems when in the following some order of integration is changed). If σ' is another representation of H such that σ is contained (as subrepresentation) in σ, then the induced imprimitivity system $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ is contained in $\left(\lambda^{\sigma'}, P^{\sigma'}, \mathcal{H}^{\sigma'}\right)$. Hence, we can always assume that σ in the previous theorem has uniform infinite multiplicity.

With this assumption, and by the equivalence $\widehat{H} \simeq \widehat{G}/H^\perp$, there exist a measure ν on $\widehat{G}/H^\perp$ and an infinite dimensional Hilbert space $\mathcal{M}$ such that, up to a unitary equivalence, σ acts diagonally on $L^2\left(\widehat{G}/H^\perp, \nu; \mathcal{M}\right)$. The first step of our construction is to diagonalise the representation $\mathrm{ind}_H^G(\sigma)$.

2.3 Diagonalisation of $\mathrm{ind}_H^G(\sigma)$

In this section, given a representation of H with uniform multiplicity, we diagonalise the corresponding induced representation.

Let ν be a measure on $\widehat{G}/H^\perp$ and $\mathcal{M}$ be a Hilbert space. Let σ^ν be the diagonal representation of H acting on the space $L^2\left(\widehat{G}/H^\perp, \nu; \mathcal{M}\right)$, that is

$$(\sigma^\nu(h)\xi)(\dot{x}) = \langle \dot{x}, h\rangle \xi(\dot{x}),$$

for all $\xi \in L^2\left(\widehat{G}/H^\perp, \nu; \mathcal{M}\right)$ and $h \in H$.

We denote by $\mathcal{H}^\nu$ the space of functions $f : G \times \widehat{G}/H^\perp \longrightarrow \mathcal{M}$ such that

1. f is weakly $(\mu_G \otimes \nu)$-measurable;

2. for all $h \in H$ and $g \in G$
$$f(gh, \dot{x}) = \overline{\langle \dot{x}, h\rangle} f(g, \dot{x}) \quad \text{for } \nu\text{-a.a. } \dot{x} \in \widehat{G}/H^\perp; \tag{2.2}$$

3. $$\int_{G/H \times \widehat{G}/H^\perp} \|f(g, \dot{x})\|^2 \, \mathrm{d}\left(\mu_{G/H} \otimes \nu\right)(\dot{g}, \dot{x}) < +\infty.$$

We identify functions in $\mathcal{H}^\nu$ that are equal $(\mu_G \otimes \nu)$-a.e.. Let G act on $\mathcal{H}^\nu$ as

$$(\lambda^\nu(a) f)(g, \dot{x}) := f\left(a^{-1}g, \dot{x}\right)$$

for all $a \in G$. Define

$$(P^\nu(\omega) f)(g, \dot{x}) := \omega(\dot{g}) f(g, \dot{x})$$

for all $f \in \mathcal{H}^\nu$, $\omega \in C_c(G/H)$.

The following proposition is stated in [21] without a proof.

Proposition 2.3.1 *The space $\mathcal{H}^\nu$ is a Hilbert space with respect to the inner product*

$$\langle f_1, f_2\rangle_{\mathcal{H}^\nu} = \int_{G/H \times \widehat{G}/H^\perp} \langle f_1(g, \dot{x}), f_2(g, \dot{x})\rangle \, d\left(\mu_{G/H} \otimes \nu\right)(\dot{g}, \dot{x}).$$

If $\varphi \in C_c\left(G \times \widehat{G}/H^\perp; \mathcal{M}\right)$, let

$$f_\varphi(g, \dot{x}) := \int_H \langle \dot{x}, h\rangle \varphi(gh, \dot{x}) \, d\mu_H(h) \qquad \forall (g, \dot{x}) \in G \times \widehat{G}/H^\perp.$$

Then f_φ is a continuous function in $\mathcal{H}^\nu$ such that $\left(\pi \times \mathrm{id}_{\widehat{G}/H^\perp}\right)(\operatorname{supp} f_\varphi)$ is compact, and the set

$$\mathcal{H}_0^\nu = \left\{ f_\varphi \mid \varphi \in C_c\left(G \times \widehat{G}/H^\perp; \mathcal{M}\right)\right\}$$

is a dense subspace of $\mathcal{H}^\nu$. The triple $(\lambda^\nu, P^\nu, \mathcal{H}^\nu)$ is the canonical imprimitivity system induced by σ^ν from H to G.

Proof. Recall the definitions of s and V given after Remark 1.2.1 in §1.2. We define a unitary operator $V' : \mathcal{H}^\nu \longrightarrow L^2\left(G/H \times \widehat{G}/H^\perp, \mu_{G/H} \otimes \nu; \mathcal{M}\right)$ by

$$\left(V'f\right)(\dot{g}, \dot{x}) = f\left(s\left(\dot{g}\right), \dot{x}\right).$$

We denote by J the canonical identification of $L^2\left(G/H \times \widehat{G}/H^\perp, \mu_{G/H} \otimes \nu; \mathcal{M}\right)$ with $L^2\left(G/H, \mu_{G/H}; L^2\left(\widehat{G}/H^\perp, \nu; \mathcal{M}\right)\right)$, where

$$\left[J\phi\left(\dot{g}\right)\right](\dot{x}) = \phi\left(\dot{g}, \dot{x}\right) \quad \text{for a.a. } \dot{x} \in \widehat{G}/H^\perp$$

for a.a. $\dot{g} \in G/H$. It is then easy to check that the unitary map $f \longmapsto \hat{f} := V^{-1}JV'f$ intertwines the imprimitivity systems $(\lambda^\nu, P^\nu, \mathcal{H}^\nu)$ and $\left(\lambda^{\sigma^\nu}, P^{\sigma^\nu}, \mathcal{H}^{\sigma^\nu}\right)$, since

$$\left[\hat{f}\left(g\right)\right](\dot{x}) = f\left(g, \dot{x}\right) \quad \text{for } \nu\text{-a.a. } \dot{x} \in \widehat{G}/H^\perp$$

for μ_G-a.a. $g \in G$.

For the second statement in the theorem, the compactness of the set $\left(\pi \times \mathrm{id}_{\widehat{G}/H^\perp}\right)(\operatorname{supp} f_\varphi)$ is clear from the definition of f_φ. Also, the continuity of f_φ is a standard consequence of dominated convergence theorem. It only remains to prove the density of $\mathcal{H}_0^\nu$. Let $\varphi_1 \in C_c(G)$, $\varphi_2 \in C_c\left(\widehat{G}/H^\perp; \mathcal{M}\right)$, and $\psi(g, x) = \varphi_1(g)\varphi_2(\dot{x})$. For $f \in \mathcal{H}^\nu$, we have

$$\begin{aligned}
\langle f, f_\psi \rangle_{\mathcal{H}^\nu} &= \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_H \mathrm{d}\mu_H(h)\, \varphi_1(gh) \\
&\quad \times \langle \langle \dot{x}, h \rangle f(g, \dot{x}), \varphi_2(\dot{x}) \rangle_{\mathcal{M}} \\
&= \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H \mathrm{d}\mu_H(h)\, \varphi_1(gh) \\
&\quad \times \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \left\langle \left[\hat{f}(g)\right](\dot{x}), \left(\sigma^\nu(h)\varphi_2\right)(\dot{x}) \right\rangle_{\mathcal{M}} \\
&= \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H \mathrm{d}\mu_H(h)\, \varphi_1(gh) \\
&\quad \times \left\langle \hat{f}(g), \sigma^\nu(h)\varphi_2 \right\rangle_{L^2\left(\widehat{G}/H^\perp, \nu; \mathcal{M}\right)} \\
&= \left\langle \hat{f}, f_{\varphi_1\varphi_2} \right\rangle_{\mathcal{H}^{\sigma^\nu}}.
\end{aligned}$$

Since the set

$$\left\{ f_{\varphi_1\varphi_2} \mid \varphi_1 \in C_c(G),\ \varphi_2 \in C_c\left(\widehat{G}/H^\perp; \mathcal{M}\right) \right\}$$

is total in $\mathcal{H}^{\sigma^\nu}$ (see Remark 1.2.5), the density of $\mathcal{H}_0^\nu$ follows. ■

We now diagonalise the representation λ^ν. First of all, we let $\widetilde{\nu}$ be the measure defined in $\widehat{G}$ by eq. (2.1). Let Λ^ν be the diagonal representation of G acting on $L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)$ as

$$\left(\Lambda^\nu(g)\,\phi\right)(x) = \langle x,g\rangle\,\phi(x) \quad \text{for a.a. } x\in\widehat{G}$$

for all $g\in G$.

Moreover, given $\phi:\widehat{G}\longrightarrow\mathcal{M}$ and fixed $x\in\widehat{G}$, define ϕ_x from $H^\perp$ to $\mathcal{M}$ as

$$\phi_x(y) := \phi(xy) \quad \forall y\in H^\perp.$$

Theorem 2.3.2 *There is a unique unitary operator Σ from $\mathcal{H}^\nu$ onto $L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)$ such that, for all $f\in\mathcal{H}_0^\nu$,*

$$(\Sigma f)(x) = \int_{G/H}\langle x,g\rangle\,f(g,\dot{x})\,\mathrm{d}\mu_{G/H}(\dot{g}) \quad \textit{for a.a. } x\in\widehat{G}. \tag{2.3}$$

The operator Σ intertwines λ^ν with Λ^ν. Moreover,

$$(\Sigma^*\varphi)(g,\dot{x}) = \int_{H^\perp}\overline{\langle xy,g\rangle}\varphi(xy)\,\mathrm{d}\mu_{H^\perp}(y) \quad \textit{for a.a. } (g,\dot{x})\in G\times\widehat{H} \tag{2.4}$$

for all $\varphi\in C_c\left(\widehat{G};\mathcal{M}\right)$.

Proof. We first define Σ on $\mathcal{H}_0^\nu$. Let $f\in\mathcal{H}_0^\nu$. Fixed $x\in\widehat{G}$, by virtue of eq. (2.2) the function

$$g\longmapsto\langle x,g\rangle\,f(g,\dot{x})$$

depends only on the equivalence class $\dot{g}$ of g and we let f^x be the corresponding map on G/H. Due to the properties of f, f^x is continuous and has compact support, so it is $\mu_{G/H}$-integrable and we define Σf by means of eq. (2.3).

We claim that Σf is in $L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)$ and $\|\Sigma f\|_{L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)} = \|f\|_{\mathcal{H}^\nu}$. Since the map

$$(x,\dot{g})\longmapsto f^x(\dot{g})$$

is continuous from $\widehat{G}\times G/H$ to $\mathcal{M}$ and has compact support, by a standard argument Σf is continuous. Moreover, if $x\in\widehat{G}$ and $y\in H^\perp$,

$$\begin{aligned}(\Sigma f)(xy) &= \int_{G/H}\langle xy,g\rangle\,f(g,\dot{x})\,\mathrm{d}\mu_{G/H}(\dot{g})\\ &= \int_{G/H}\langle y,\dot{g}\rangle\,\langle x,g\rangle\,f(g,\dot{x})\,\mathrm{d}\mu_{G/H}(\dot{g})\\ &= \overline{\mathcal{F}}_{G/H}(f^x)(y).\end{aligned}$$

We then have

$$
\begin{aligned}
\|\Sigma f\|^2_{L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)} &= \int_{\widehat{G}} \|(\Sigma f)(x)\|^2 \,\mathrm{d}\widetilde{\nu}(x) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \|(\Sigma f)(xy)\|^2 \,\mathrm{d}\mu_{H^\perp}(y) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \left\|\overline{\mathcal{F}}_{G/H}(f^x)(y)\right\|^2 \mathrm{d}\mu_{H^\perp}(y) \\
&\quad \text{(unitarity of } \overline{\mathcal{F}}_{G/H}\text{)} \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \|f^x(\dot{g})\|^2 \,\mathrm{d}\mu_{G/H}(\dot{g}) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \|f(g,\dot{x})\|^2 \,\mathrm{d}\mu_{G/H}(\dot{g}) \\
&= \int_{G/H\times\widehat{G}/H^\perp} \|f(g,\dot{x})\|^2 \,\mathrm{d}\left(\mu_{G/H}\otimes\nu\right)(\dot{g},\dot{x}) \\
&= \|f\|^2_{\mathcal{H}^\nu}.
\end{aligned}
$$

By density, Σ extends to an isometry from $\mathcal{H}^\nu$ to $L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)$. Clearly, eq. (2.3) holds and it defines uniquely Σ.

The second step is computing the adjoint of Σ. Let $\varphi \in C_c\left(\widehat{G};\mathcal{M}\right)$, by standard arguments the right hand side of eq. (2.4) is a continuous function of $(g,\dot{x})$. Moreover, it satisfies eq. (2.2). We have

$$
\int_{H^\perp} \overline{\langle xy, g\rangle}\varphi(xy)\,\mathrm{d}\mu_{H^\perp}(y) = \overline{\langle x, g\rangle}\overline{\mathcal{F}}^*_{G/H}(\varphi_x)(\dot{g}).
$$

First of all, we show that the above function of $(g,\dot{x})$ is in $\mathcal{H}^\nu$. Indeed,

$$
\begin{aligned}
&\int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \left\|\overline{\langle x, g\rangle}\overline{\mathcal{F}}^*_{G/H}(\varphi_x)(\dot{g})\right\|^2 \mathrm{d}\mu_{G/H}(\dot{g}) \\
&\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \left\|\overline{\mathcal{F}}^*_{G/H}(\varphi_x)(\dot{g})\right\|^2 \mathrm{d}\mu_{G/H}(\dot{g}) \\
&\quad\quad \text{(unitarity of } \overline{\mathcal{F}}_{G/H}\text{)} \\
&\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \|\varphi_x(y)\|^2 \,\mathrm{d}\mu_{H^\perp}(y) \\
&\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \|\varphi(xy)\|^2 \,\mathrm{d}\mu_{H^\perp}(y) \\
&\quad = \|\varphi\|^2_{L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)}. \qquad (2.5)
\end{aligned}
$$

Moreover, for all $f \in \mathcal{H}_0^\nu$, we have

$$
\begin{aligned}
\langle f, \Sigma^* \varphi \rangle_{\mathcal{H}^\nu} =& \langle \Sigma f, \varphi \rangle_{L^2(\widehat{G}, \widetilde{\nu}; \mathcal{M})} \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{H^\perp} \left\langle \left(\Sigma f\right)(xy), \varphi\left(xy\right)\right\rangle \mathrm{d}\mu_{H^\perp}\left(y\right) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{H^\perp} \left\langle \overline{\mathcal{F}}_{G/H}\left(f^x\right)(y), \varphi_x\left(y\right)\right\rangle \ \mathrm{d}\mu_{H^\perp}\left(y\right) \\
&\quad \text{(unitarity of } \overline{\mathcal{F}}_{G/H}\text{)} \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{G/H} \left\langle f^x\left(\dot{g}\right), \overline{\mathcal{F}}^*_{G/H}\left(\varphi_x\right)\left(\dot{g}\right)\right\rangle \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{G/H} \left\langle \langle x, g\rangle f\left(g, \dot{x}\right), \overline{\mathcal{F}}^*_{G/H}\left(\varphi_x\right)\left(\dot{g}\right)\right\rangle \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&= \int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{G/H} \left\langle f\left(g, \dot{x}\right), \overline{\langle x, g\rangle}\overline{\mathcal{F}}^*_{G/H}\left(\varphi_x\right)\left(\dot{g}\right)\right\rangle \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&= \int_{G/H \times \widehat{G}/H^\perp} \left\langle f\left(g, \dot{x}\right), \overline{\langle x, g\rangle}\overline{\mathcal{F}}^*_{G/H}\left(\varphi_x\right)\left(\dot{g}\right)\right\rangle \mathrm{d}\left(\mu_{G/H} \otimes \nu\right)\left(\dot{g}, \dot{x}\right).
\end{aligned}
$$

Since $\mathcal{H}_0^\nu$ is dense, eq. (2.4) follows. By eq. (2.5) Σ^* is isometric, hence Σ is unitary.

Finally, we show the intertwining property. Let $a \in G$ and $f \in \mathcal{H}_0^\nu$. Then $\lambda^\nu(a) f \in \mathcal{H}_0^\nu$, and so one has

$$
\begin{aligned}
\left(\Sigma \lambda^\nu(a) f\right)(x) &= \int_{G/H} \langle x, g\rangle f\left(a^{-1} g, \dot{x}\right) \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&= \langle x, a\rangle \int_{G/H} f^x\left(a^{-1}[\dot{g}]\right) \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&\quad \left(\dot{g} \longrightarrow a\left[\dot{g}\right]\right) \\
&= \langle x, a\rangle \int_{G/H} \langle x, g\rangle f\left(g, \dot{x}\right) \mathrm{d}\mu_{G/H}\left(\dot{g}\right) \\
&= \left(\Lambda^\nu(a) \Sigma f\right)(x).
\end{aligned}
$$

By density of $\mathcal{H}_0^\nu$, it follows that $\Sigma \lambda^\nu(a) = \Lambda^\nu(a) \Sigma$. ■

Given $\omega \in C_c(G/H)$, let $\widetilde{P^\nu}(\omega) = \Sigma P^\nu(\omega) \Sigma^*$. Then

Proposition 2.3.3 *For all $\omega \in C_c(G/H)$ and $\phi \in L^2\left(\widehat{G}, \widetilde{\nu}; \mathcal{M}\right)$,*

$$
\left(\widetilde{P^\nu}(\omega) \phi\right)(x) = \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \phi\left(xy^{-1}\right) d\mu_{H^\perp}(y) \quad \textit{for a.a. } x \in \widehat{G}. \tag{2.6}
$$

Proof. Let $\omega \in C_c(G/H)$. We compute the action of $\widetilde{P^\nu}(\omega)$ on $C_c\left(\widehat{G};\mathcal{M}\right)$. If $\varphi \in C_c\left(\widehat{G};\mathcal{M}\right)$, let

$$\xi(x) := \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y)\,\varphi\left(xy^{-1}\right) \mathrm{d}\mu_{H^\perp}(y) \qquad \forall x \in \widehat{G},$$

which is well defined and continuous. Moreover, for all $x \in \widehat{G}$ and $y \in H^\perp$,

$$\begin{aligned} \xi(xy) &= \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y')\,\varphi\left(xyy'^{-1}\right) \mathrm{d}\mu_{H^\perp}(y') \\ &= \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y')\,\varphi_x\left(yy'^{-1}\right) \mathrm{d}\mu_{H^\perp}(y') \\ &= \left(\overline{\mathcal{F}}_{G/H}(\omega) * \varphi_x\right)(y). \end{aligned} \tag{2.7}$$

Here and in the following, convolutions are always taken in $H^\perp$. If $\varphi, \psi \in C_c\left(\widehat{G};\mathcal{M}\right)$,

$$\begin{aligned} &\left\langle \psi, \widetilde{P^\nu}(\omega)\varphi \right\rangle_{L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)} = \left\langle \Sigma^*\psi, P^\nu(\omega)\,\Sigma^*\varphi\right\rangle_{\mathcal{H}^\nu} \\ &\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \Big\langle (\dot{g})\,\overline{\langle x,g\rangle}\overline{\mathcal{F}}^*_{G/H}(\psi_x)(\dot{g}), \omega \\ &\qquad \times \overline{\langle x,g\rangle}\overline{\mathcal{F}}^*_{G/H}(\varphi_x)(\dot{g}) \Big\rangle \\ &\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \left\langle (\dot{g})\,\overline{\mathcal{F}}^*_{G/H}(\psi_x)(\dot{g}), \omega\overline{\mathcal{F}}^*_{G/H}(\varphi_x)(\dot{g}) \right\rangle \\ &\qquad \text{(unitarity of } \overline{\mathcal{F}}_{G/H} \text{ and properties of convolution)} \\ &\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \mathrm{d}\mu_{H^\perp}(y) \left\langle \psi_x(y), \left(\overline{\mathcal{F}}_{G/H}(\omega) * \varphi_x\right)(y)\right\rangle \\ &\quad = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \mathrm{d}\mu_{H^\perp}(y) \left\langle \psi(xy), \xi(xy)\right\rangle, \end{aligned}$$

hence eq. (2.6) holds on $C_c\left(\widehat{G};\mathcal{M}\right)$.

Let now $\phi \in L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)$. Since

$$\|\phi\|^2_{L^2\left(\widehat{G},\widetilde{\nu};\mathcal{M}\right)} = \int_{\widehat{G}/H^\perp} \mathrm{d}\nu(\dot{x}) \int_{H^\perp} \|\phi(xy)\|^2 \mathrm{d}\mu_{H^\perp}(y) < +\infty,$$

by virtue of Fubini's theorem there is a ν-negligible set $X_1 \subset \widehat{G}/H^\perp$ such that, for all $x \in \widehat{G}$ with $\dot{x} \notin X_1$, $\phi_x \in L^2\left(H^\perp, \mu_{H^\perp};\mathcal{M}\right)$. Moreover, using the definition of $\widetilde{\nu}$, one can check that $q^{-1}(X_1)$ is $\widetilde{\nu}$-negligible. Then, for $\widetilde{\nu}$-almost all $x \in \widehat{G}$, ϕ_x is in $L^2\left(H^\perp, \mu_{H^\perp};\mathcal{M}\right)$. We observe that the map

$$\dot{g} \longmapsto \omega(\dot{g})\left(\overline{\mathcal{F}}^*_{G/H}(\phi_x)\right)(\dot{g})$$

is then in $\left(L^1 \cap L^2\right)\left(G/H, \mu_{G/H}; \mathcal{M}\right)$ for $\widetilde{\nu}$-almost all $x \in \widehat{G}$, hence its Fourier cotransform is continuous, and we have

$$\begin{aligned}
\overline{\mathcal{F}}_{G/H}\left(\omega \overline{\mathcal{F}}^*_{G/H}\left(\phi_x\right)\right)(e) &= \left(\overline{\mathcal{F}}_{G/H}\left(\omega\right) * \phi_x\right)(e) \\
&= \int_{H^\perp} \overline{\mathcal{F}}_{G/H}\left(\omega\right)(y)\, \phi\left(xy^{-1}\right) \mathrm{d}\mu_{H^\perp}(y) \quad (2.8)
\end{aligned}$$

Now, we let $\left(\varphi_k\right)_{k \geq 1}$ be a sequence in $C_c\left(\widehat{G}; \mathcal{M}\right)$ converging to ϕ in $L^2\left(\widehat{G}, \widetilde{\nu}; \mathcal{M}\right)$. Then

$$\int_{\widehat{G}/H^\perp} \mathrm{d}\nu\left(\dot{x}\right) \int_{H^\perp} \left\|\left(\varphi_k\right)_x(y) - \phi_x(y)\right\|^2 \mathrm{d}\mu_{H^\perp}(y) \longrightarrow 0$$

and so, possibly passing to a subsequence, there is a ν-negligible set $X_2 \subset \widehat{G}/H^\perp$ such that

$$\int_{H^\perp} \left\|\left(\varphi_k\right)_x(y) - \phi_x(y)\right\|^2 \mathrm{d}\mu_{H^\perp}(y) \longrightarrow 0$$

for all $x \in \widehat{G}$ with $\dot{x} \notin X_2$. This fact means that, for $\widetilde{\nu}$-almost all $x \in \widehat{G}$,

$$\left(\varphi_k\right)_x \longrightarrow \phi_x$$

in $L^2\left(H^\perp, \mu_{H^\perp}; \mathcal{M}\right)$. It follows that

$$\omega \overline{\mathcal{F}}^*_{G/H}\left(\left(\varphi_k\right)_x\right) \longrightarrow \omega \overline{\mathcal{F}}^*_{G/H}\left(\phi_x\right)$$

in $L^1\left(G/H, \mu_{G/H}; \mathcal{M}\right)$. Then, for $\widetilde{\nu}$-almost all $x \in \widehat{G}$,

$$\overline{\mathcal{F}}_{G/H}\left(\omega \overline{\mathcal{F}}^*_{G/H}\left(\left(\varphi_k\right)_x\right)\right) \longrightarrow \overline{\mathcal{F}}_{G/H}\left(\omega \overline{\mathcal{F}}^*_{G/H}\left(\phi_x\right)\right)$$

uniformly, and, using eqs. (2.7), (2.8),

$$\begin{aligned}
&\left(\widetilde{P^\nu}\left(\omega\right) \varphi_k\right)(x) = \overline{\mathcal{F}}_{G/H}\left(\omega \overline{\mathcal{F}}^*_{G/H}\left(\left(\varphi_k\right)_x\right)\right)(e) \longrightarrow \\
&\qquad \longrightarrow \overline{\mathcal{F}}_{G/H}\left(\omega \overline{\mathcal{F}}^*_{G/H}\left(\phi_x\right)\right)(e) = \int_{H^\perp} \overline{\mathcal{F}}_{G/H}\left(\omega\right)(y)\, \phi\left(xy^{-1}\right) \mathrm{d}\mu_{H^\perp}(y).
\end{aligned}$$

Since $\widetilde{P^\nu}\left(\omega\right) \varphi_k$ converges to $\widetilde{P^\nu}\left(\omega\right) \phi$ in $L^2\left(\widehat{G}, \widetilde{\nu}; \mathcal{M}\right)$, eq. (2.6) follows from uniqueness of the limit. ■

2.4 Characterisation of covariant POM's

We fix in the following an *infinite dimensional* Hilbert space $\mathcal{M}$. According to the results of the previous sections, theorem 2.2.2 can be stated in the following way.

Theorem 2.4.1 *A POM E based on G/H and acting on $\mathcal{H}$ is covariant with respect to U if and only if there exist a measure ν on $\widehat{G}/H^{\perp}$ and an isometry W intertwining U with Λ^{ν} such that*

$$E(\omega) = W^{*}\widetilde{P^{\nu}}(\omega)W$$

for all $\omega \in C_c(G/H)$.

To get an explicit form of W, we assume that U acts diagonally on $\mathcal{H}$. This means that $\mathcal{H}$ is the orthogonal sum of invariant subspaces

$$\mathcal{H} = \bigoplus_{k\in I} L^2\left(\widehat{G}, \rho_k; F_k\right), \tag{2.9}$$

where I is a denumerable set, $(\rho_k)_{k\in I}$ is a family of *pairwise disjoint* measures on $\widehat{G}$, $(F_k)_{k\in I}$ is a family of Hilbert spaces, and the action of U is given by

$$(U(g)\phi_k)(x) = \langle x, g\rangle \phi_k(x) \qquad x \in \widehat{G},$$

where $\phi_k \in L^2\left(\widehat{G}, \rho_k; F_k\right)$ and $g \in G$. We will denote by P_k the orthogonal projector onto the invariant subspace $L^2\left(\widehat{G}, \rho_k; F_k\right)$.

The assumption (2.9) is not restrictive. Indeed, by Theorem 2.2.1 there are a family of disjoint measures $(\rho_k)_{k\in\mathbb{Z}_+\cup\{\infty\}}$ and a family of Hilbert spaces $(F_k)_{k\in\mathbb{Z}_+\cup\{\infty\}}$ such that $\dim F_k = k$ and, up to unitary equivalence, eq. (2.9) holds.

Given the decomposition (2.9), let ρ be a measure on $\widehat{G}$ such that

$$\rho(N) = 0 \Longleftrightarrow \rho_k(N) = 0 \quad \forall k \in I. \tag{2.10}$$

We recall that the equivalence class of ρ is uniquely defined by the family $(\rho_k)_{k\in I}$.

The following proposition was stated in [21] without a proof.

Proposition 2.4.2 *The equivalence class of the measure ρ defined in eq. (2.10) is independent of the choice of decomposition (2.9).*

Proof. Fix a decomposition of $\mathcal{H}$ as in eq. (2.9). For $j \in \mathbb{Z}_+ \cup \{\infty\}$, let $I_j = \{k \in I \mid \dim F_k = j\}$. Let ρ'_j be a measure on $\widehat{G}$ such that

$$\rho'_j(N) = 0 \Longleftrightarrow \rho_k(N) = 0 \quad \forall k \in I_j.$$

Fix Hilbert spaces $\left(F'_j\right)_{j\in\mathbb{Z}_+\cup\{\infty\}}$ such that $\dim F'_j = j$. Then we can estabilish a unitary equivalence V between U and the diagonal representation U' acting in $\mathcal{H}' = \bigoplus_{j\in\mathbb{Z}_+\cup\{\infty\}} L^2\left(\widehat{G}, \rho'_j; F'_j\right)$. Namely, for $k \in I_j$, let $\gamma_{k,j}$ be the density of ρ_k with respect to ρ'_j, and $V_k : F_k \longrightarrow F'_j$ be a fixed unitary map. Then we set

$$\left(P'_j V P_k \phi\right)(x) = \sqrt{\gamma_{k,j}(x)} V_k \left(P_k \phi\right)(x),$$

where P'_j is the orthogonal projector onto the invariant subspace $L^2\left(\widehat{G}, \rho'_j; F'_j\right)$ of $\mathcal{H}'$. By the definitions we have

$$\rho(N) = 0 \Longleftrightarrow \rho'_j(N) = 0 \quad \forall j \in \mathbb{Z}_+ \cup \{\infty\}.$$

Since by Theorem 2.2.1 the equivalence class of each ρ'_j is uniquely determined by the representation U, the equivalence class of ρ is independent of the choice of decomposition (2.9). ■

By the above proposition, the representation U defines uniquely an equivalence class $\mathcal{C}_U$ of measures ρ such that relation (2.10) holds. Choosing in this equivalence class a *finite* measure ρ, we denote by $\mathcal{C}_U^q$ the equivalence class of the image measure ρ^q on $\widehat{G}/H^\perp$. Clearly $\mathcal{C}_U^q$ depends only on $\mathcal{C}_U$.

We now give the central result of this chapter (and of [21]).

Theorem 2.4.3 *Let U be a representation of G acting diagonally on the space $\mathcal{H}$ of eq. (2.9). Given $\nu_U \in \mathcal{C}_U^q$, let $\widetilde{\nu}_U$ be the measure given by eq. (2.1). The representation U admits covariant positive operator valued measures based on G/H if and only if, for all $k \in I$, ρ_k has density with respect to $\widetilde{\nu}_U$. In this case, for every $k \in I$, let α_k be the density of ρ_k with respect to $\widetilde{\nu}_U$.*

Let $\mathcal{M}$ be a fixed infinite dimensional Hilbert space. For each $k \in I$, let

$$\widehat{G} \ni x \longmapsto W_k(x) \in \mathcal{L}(F_k; \mathcal{M})$$

be a weakly measurable map such that $W_k(x)$ are isometries for ρ_k-almost all $x \in \widehat{G}$. For $\omega \in C_c(G/H)$, let $E(\omega)$ be the operator whose action on $\phi \in \mathcal{H}$ is given by

$$\begin{aligned}\left(P_j E(\omega) P_k \phi\right)(x) &= \int_{H^\perp} d\mu_{H^\perp}(y) \overline{\mathcal{F}}_{G/H}(\omega)(y) \sqrt{\frac{\alpha_k\left(xy^{-1}\right)}{\alpha_j(x)}} \\ &\quad \times W_j(x)^* W_k\left(xy^{-1}\right)\left(P_k\phi\right)\left(xy^{-1}\right) \end{aligned} \tag{2.11}$$

for ρ_j-almost all $x \in \widehat{G}$ and $k, j \in I$. Then, E is a POM covariant with respect to U.

Conversely, any POM based on G/H and covariant with respect to U is of the form given by eq. (2.11).

We add some comments before the proof of the theorem.

Remark 2.4.4 *We observe that eq. (2.11) is invariant with respect to the choice of the measure* $\nu_U \in \mathcal{C}_U^q$*. Indeed, let* $\nu'_U \in \mathcal{C}_U^q$*, and* $\beta > 0$ *be the density of* ν_U *with respect to* ν'_U*. Clearly*

$$\widetilde{\nu_U} = (\beta \circ q)\, \widetilde{\nu'_U},$$

so that the density α'_k *of* ρ_k *with respect to* $\widetilde{\nu'_U}$ *is*

$$\alpha'_k = (\beta \circ q)\, \alpha_k.$$

It follows that eq. (2.11) does not depend on the choice of $\nu_U \in \mathcal{C}_U^q$*.*

Corollary 2.4.5 *Let* H *be the trivial subgroup* $\{e\}$*. The representation* U *admits covariant positive operator valued measures based on* G *if and only if the measures* ρ_k *have density with respect to the Haar measure* $\mu_{\widehat{G}}$*. In this case, the functions* α_k *in eq. (2.11) are the densities of* ρ_k *with respect to* $\mu_{\widehat{G}}$*.*

Remark 2.4.6 *The content of the previous corollary was first shown by Holevo in ref. [41] for non-normalised POM. In order to compare the two results observe that, if* $\phi \in \left(L^1 \cap L^2\right)\left(\widehat{G}, \rho_k; F_k\right)$ *and* $\psi \in \left(L^1 \cap L^2\right)\left(\widehat{G}, \rho_j; F_j\right)$*, eq. (2.11) becomes*

$$\begin{aligned}\langle \phi, E(\omega)\, \psi \rangle_{\mathcal{H}} &= \int_G d\mu_G(g)\, \omega(g) \int_{\widehat{G}\times\widehat{G}} \langle y, g\rangle\, \overline{\langle x, g\rangle} \sqrt{\alpha_k(y)\, \alpha_j(x)} \\ &\quad \times \langle W_j(x)^* W_k(y)\, \phi(y), \psi(x)\rangle\, d\left(\mu_{\widehat{G}} \otimes \mu_{\widehat{G}}\right)(x, y) \\ &= \int_G d\mu_G(g)\, \omega(g) \int_{\widehat{G}\times\widehat{G}} K_{U(g^{-1})\psi, U(g^{-1})\phi}(x, y)\, d\left(\mu_{\widehat{G}} \otimes \mu_{\widehat{G}}\right)(x, y),\end{aligned}$$

where

$$K_{\psi,\phi}(x, y) = \sqrt{\alpha_k(y)\, \alpha_j(x)}\, \langle W_k(y)\, \phi(y), W_j(x)\, \psi(x)\rangle$$

is a bounded positive definite measurable field of forms (compare with eqs. (4.2) *and* (4.3) *in ref. [41]).*

In order to prove Theorem 2.4.3, we need the following lemma.

Lemma 2.4.7 *Let* ρ *be a finite measure on* $\widehat{G}$*. Assume that there is a measure* ν *on* $\widehat{G}/H^\perp$ *such that* ρ *has density with respect to* $\widetilde{\nu}$*. Then* ρ *has density with respect to* $\widetilde{\rho^q}$*. In this case,* ν *uniquely decomposes as*

$$\nu = \nu_1 + \nu_2,$$

where ν_1 *is equivalent to* ρ^q *and* $\nu_2 \perp \rho^q$*.*

Proof. Suppose that ν is a measure on $\widehat{G}/H^{\perp}$ such that $\rho = \alpha\widetilde{\nu}$, where α is a non-negative $\widetilde{\nu}$-integrable function on $\widehat{G}$. Then, for all $\varphi \in C_c\left(\widehat{G}/H^{\perp}\right)$,

$$\begin{aligned}\rho^q(\varphi) &= \int_{\widehat{G}} \varphi(q(x))\,\mathrm{d}\rho(\dot{x}) \\ &= \int_{\widehat{G}/H^{\perp}} \mathrm{d}\nu(\dot{x}) \int_{H^{\perp}} \varphi(\dot{x})\,\alpha(xy)\,\mathrm{d}\mu_{H^{\perp}}(y) \\ &= \int_{\widehat{G}/H^{\perp}} \varphi(\dot{x})\,\alpha'(\dot{x})\,\mathrm{d}\nu(\dot{x}),\end{aligned}$$

where the function

$$\alpha'(\dot{x}) := \int_{H^{\perp}} \alpha(xy)\,\mathrm{d}\mu_{H^{\perp}}(y) \geq 0$$

is ν-integrable by virtue of Fubini theorem. It follows that

$$\rho^q = \alpha'\nu. \tag{2.12}$$

Using Lebesgue theorem, we can uniquely decompose

$$\nu = \nu_1 + \nu_2,$$

where ν_1 has base ρ^q and $\nu_2 \perp \rho^q$. From eq. (2.12), it follows that ν_1 and ρ^q are equivalent, and this proves the second statement of the lemma. If $A, B \in \mathcal{B}\left(\widehat{G}/H^{\perp}\right)$ are disjoint sets such that ν_2 is concentrated in A and ν_1 is concentrated in B, then $\widetilde{\nu_2}$ and $\widetilde{\nu_1}$ are respectively concentrated in the disjoint sets $\widetilde{A} = q^{-1}(A)$ and $\widetilde{B} = q^{-1}(B)$. By definition of ρ^q, we also have

$$\rho\left(\widetilde{A}\right) = \rho^q(A) = 0.$$

Since ρ has density with respect to $\widetilde{\nu} = \widetilde{\nu_1} + \widetilde{\nu_2}$ and $\widetilde{\nu_2}$ is concentrated in $\widetilde{A}$, it follows that ρ has density with respect to $\widetilde{\nu_1} \cong \widetilde{\rho^q}$. The claim is now clear. ■

Proof of Theorem 2.4.3. Let ρ be a finite measure in $\mathcal{C}_U$. By virtue of Theorem 2.4.1, U admits a covariant POM $\Longleftrightarrow$ there exists a measure ν in $\widehat{G}/H^{\perp}$ such that U is a subrepresentation of Λ^{ν} $\Longleftrightarrow$ each measure ρ_k has density with respect to $\widetilde{\nu}$ $\Longleftrightarrow$ ρ has density with respect to $\widetilde{\nu}$. From Lemma 2.4.7, U admits a covariant POM if and only if ρ has density with respect to $\widetilde{\rho^q}$. Since $\rho^q \in \mathcal{C}_U^q$, the first claim follows.

Let now E be a covariant POM. By Theorem 2.4.1, there is a measure ν on $\widehat{G}/H^{\perp}$ and an isometry W intertwining U with Λ^{ν} such that

$$E(\omega) = W^* \widetilde{P^{\nu}}(\omega)\, W \qquad \forall \omega \in C_c(G/H).$$

Using Lemma 2.4.7, we (uniquely) decompose

$$\nu = \nu_1 + \nu_2,$$

where ν_1 is equivalent to ν_U and $\nu_2 \perp \nu_U$. Then we have

$$\sigma^\nu \simeq \sigma^{\nu_U} \oplus \sigma^{\nu_2},$$

from which the decomposition of the corresponding induced imprimitivity systems follows:

$$\begin{aligned}\left(\Lambda^\nu, \widetilde{P^\nu}, L^2\left(\widehat{G}, \widetilde{\nu}; \mathcal{M}\right)\right) \quad &\simeq \quad \left(\Lambda^{\nu_U}, \widetilde{P^{\nu_U}}, L^2\left(\widehat{G}, \widetilde{\nu_U}; \mathcal{M}\right)\right) \\ &\quad \oplus \left(\Lambda^{\nu_2}, \widetilde{P^{\nu_2}}, L^2\left(\widehat{G}, \widetilde{\nu_2}; \mathcal{M}\right)\right).\end{aligned}$$

Moreover, since each ρ_k has density with respect to $\widetilde{\nu_U}$ and $\widetilde{\nu_U}$ is disjoint from $\widetilde{\nu_2}$, it follows that $W\mathcal{H} \subset L^2\left(\widehat{G}, \widetilde{\nu_U}; \mathcal{M}\right)$. Thus, we can always assume that the measure ν on $\widehat{G}/H^\perp$ which occurs in Theorem 2.4.1 is ν_U.

We now characherise the form of W. For $k \in I$, we can always fix an isometry $T_k : F_k \longrightarrow \mathcal{M}$ such that $T_k(F_k)$ are mutually orthogonal subspaces of $\mathcal{M}$. Hence, if we define, for $\phi_k \in L^2\left(\widehat{G}, \rho_k; F_k\right)$,

$$(T\phi_k)(x) := \sqrt{\alpha_k(x)} T_k \phi_k(x),$$

T is an isometry intertwining U with Λ^{ν_U}. We define $W_k = WP_k$. The operator $V = WT^*$ is a partial isometry commuting with Λ^{ν_U}, hence there exists a weakly measurable correspondence $\widehat{G} \ni x \longmapsto V(x) \in \mathcal{L}(\mathcal{M})$ such that $V(x)$ are partial isometries for $\widetilde{\nu_U}$-almost all $x \in \widehat{G}$ and

$$(V\phi)(x) = V(x)\phi(x) \quad \text{for a.a. } x \in \widehat{G},$$

where $\phi \in L^2\left(\widehat{G}, \widetilde{\nu_U}; \mathcal{M}\right)$. We have $W = WT^*T = VT$, then

$$\begin{aligned}(W_k\phi_k)(x) &= \sqrt{\alpha_k(x)} V(x) T_k \phi_k(x) \\ &= \sqrt{\alpha_k(x)} W_k(x) \phi_k(x),\end{aligned} \tag{2.13}$$

where we set

$$W_k(x) = V(x) T_k \quad \forall x \in \widehat{G}.$$

Since W is isometric, then $W_k^* W_k$ is the identity operator on $L^2\left(\widehat{G}, \rho_k; F_k\right)$, hence

$$T_k^* V(x)^* V(x) T_k = I_k$$

for ρ_k-almost all $x \in \widehat{G}$, where I_k is the identity operator on F_k. Since T_k is isometric and $V(x)$ is a partial isometry for $\widetilde{\nu_U}$-almost every $x \in \widehat{G}$ (that is for ρ_k-almost every $x \in \widehat{G}$), it follows that $V(x)^* V(x)$ is the identity on $\operatorname{ran} T_k$ and that $W_k(x)$ is isometric, for ρ_k-almost every $x \in \widehat{G}$. Weak measurability of the maps $x \longmapsto W_k(x)$ is immediate.

The explicit form of E is then given by

$$\begin{aligned}(P_j E(\omega) P_k \phi)(x) &= \left(W_j^* \widetilde{P^\nu}(\omega) W_k \phi\right)(x) \\ &= \frac{1}{\sqrt{\alpha_j(x)}} W_j(x)^* \int_{H^\perp} \overline{\mathcal{F}}_{G/H}(\omega)(y) \\ &\quad \times \sqrt{\alpha_k(xy^{-1})} W_k\left(xy^{-1}\right)(P_k\phi)\left(xy^{-1}\right) \mathrm{d}\mu_{H^\perp}(y),\end{aligned}$$

where $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$.

Conversely, let $\widehat{G} \ni x \longmapsto W_k(x) \in \mathcal{L}(F_k; \mathcal{M})$ be a weakly measurable map such that $W_k(x)$ are isometries for ρ_k-almost every $x \in \widehat{G}$ and for all $k \in I$. We define, for $\phi_k \in L^2\left(\widehat{G}, \rho_k; F_k\right)$,

$$(W\phi_k)(x) := \sqrt{\alpha_k(x)} W_k(x) \phi_k(x),$$

then W is clearly an intertwining isometry between U and Λ^{ν_U} and eq. (2.11) defines a covariant POM. ■

We now study the problem of equivalence of covariant POM's.

Let E and E' be two covariant positive operator valued measures that are equivalent, i.e. there exists an unitary operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ such that

$$SU(g) = U(g) S \quad \forall g \in G, \tag{2.14}$$

$$SE(\omega) = E'(\omega) S \quad \forall \omega \in C_c(G/H). \tag{2.15}$$

We have the following result.

Proposition 2.4.8 *Let $(W_j)_{j\in I}$ and $\left(W_j'\right)_{j\in I}$ be families of maps such that eq. (2.11) holds for E and E', respectively.*

The POM's E and E' are equivalent if and only if, for each $k \in I$, there exists a weakly measurable map $x \longmapsto S_k(x) \in \mathcal{L}(F_k)$ such that $S_k(x)$ are unitary operators for ρ_k-almost all x and

$$\sqrt{\alpha_k(xy)} W_j(x)^* W_k(xy) = \sqrt{\alpha_k(xy)} S_j(x)^* W_j'(x)^* W_k'(xy) S_k(xy) \tag{2.16}$$

for $(\rho_j \otimes \mu_{H^\perp})$-almost all (x, y).

Proof. By virtue of condition (2.14) and orthogonality of the measures ρ_k, S preserves decomposition (2.9). Moreover, for each $k \in I$, there exists a weakly measurable map $x \longmapsto S_k(x) \in \mathcal{L}(F_k)$ such that $S_k(x)$ is unitary for ρ_k-almost all x and, if $\phi_k \in L^2\left(\widehat{G}, \rho_k; F_k\right)$,

$$(S\phi_k)(x) = S_k(x) \phi_k(x) \quad \text{for a.a. } x \in \widehat{G}.$$

Condition (2.15) is equivalent to

$$P_j E(\omega) P_k \phi = P_j S^* E'(\omega) S P_k \phi$$

for all $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$ and $j, k \in I$. It is not restrictive to assume that the densities α_k are measurable functions. Let

$$\Omega_{j,k}(x, x') = \sqrt{\frac{\alpha_k(x')}{\alpha_j(x)}} \left(W_j(x)^* W_k(x') - S_j(x)^* W_j'(x)^* W_k'(x') S_k(x') \right),$$

using eq. (2.11), the previous condition becomes

$$\int_{H^\perp} \mathcal{F}_{G/H}(\omega)(y)\, \Omega_{j,k}\left(x, xy^{-1}\right) (P_k \phi)\left(xy^{-1}\right) \mathrm{d}\mu_{H^\perp}(y) = 0 \qquad (2.17)$$

ρ_j-almost everywhere for all $\phi \in \mathcal{H}$, $\omega \in C_c(G/H)$ and $j, k \in I$.

Let K be a compact set of $\widehat{G}$ and $v \in F_k$. In eq. (2.17) we choose

$$\phi = \chi_K\, v \in L^2\left(\widehat{G}, \rho_k; F_k\right)$$

and $\omega \in C_c(G/H)$ running over a denumerable subset dense in $L^2(G/H, \mu_{H^\perp})$. It follows that there exists a ρ_j-null set $N \subset \widehat{G}$ such that, for all $x \notin N$,

$$\chi_K\left(xy^{-1}\right) \Omega_{j,k}\left(x, xy^{-1}\right) v = 0$$

for $\mu_{H^\perp}$-almost all $y \in H^\perp$. Since $\Omega_{j,k}$ is weakly measurable, the last equation holds in a measurable subset $X \subset \widehat{G} \times H^\perp$ whose complement is a $(\rho_j \otimes \mu_{H^\perp})$-null set. Define

$$m(x, y) = xy^{-1} \qquad \forall (x, y) \in \widehat{G} \times H^\perp.$$

For all $(x, y) \in X \cap m^{-1}(K)$ we then have

$$\Omega_{j,k}\left(x, xy^{-1}\right) v = 0.$$

Since F_k is separable and $\widehat{G}$ is σ-compact, we get

$$\Omega_{j,k}(x, xy) = 0$$

for $(\rho_j \otimes \mu_{H^\perp})$-almost all $(x, y) \in \widehat{G} \times H^\perp$, that is,

$$\sqrt{\alpha_k(xy)} W_j(x)^* W_k(xy) = \sqrt{\alpha_k(xy)} S_j(x)^* W_j'(x)^* W_k'(xy) S_k(xy)$$

for $(\rho_j \otimes \mu_{H^\perp})$-almost all (x, y).

Conversely, if condition (2.16) is satisfied for all $j, k \in I$, then clearly E is equivalent to E'. ■

2.5 Examples

2.5.1 Translation covariant observables

Let $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. We consider the representation U of the group $\mathbb{R}$ acting on $\mathcal{H}$ as

$$(U(a)\phi)(x) = e^{iax}\phi(x) \qquad x \in \mathbb{R}$$

for all $a \in \mathbb{R}$. By means of Fourier transform $\overline{\mathcal{F}}^*_{\mathbb{R}}$, U is clearly equivalent to the regular representation of $\mathbb{R}$. We classify the POM's based on $\mathbb{R}$ and covariant with respect to U. With the notations of the previous sections, we have

$$G = \mathbb{R}, \quad H = \{0\}, \quad G/H = \mathbb{R}, \quad \widehat{G} = H^\perp = \mathbb{R}, \quad \widehat{G}/H^\perp = \{0\}.$$

We choose $\mu_{G/H} = \frac{1}{2\pi}\mathrm{d}x$, so that $\mu_{H^\perp} = \mathrm{d}x$, and $\mathcal{M} = \mathcal{H}$.

The representation U is already diagonal with multiplicity equal to 1, so that in the decomposition (2.9) we can set $I = \{1\}$, $\rho_1 = \mathrm{d}x$, $F_1 = \mathbb{C}$. Hence, by Corollary 2.4.5, U admits covariant POM's based on $\mathbb{R}$ and $\alpha_1 = 1$.

According to Theorem 2.4.3, any covariant POM E is defined in terms of a weakly measurable map $x \longmapsto W_1(x)$ such that $W_1(x) : \mathbb{C} \longrightarrow \mathcal{H}$ is an isometry for every $x \in \mathbb{R}$. This is equivalent to selecting a weakly measurable map $x \longmapsto h_x \in \mathcal{H}$, with $\|h_x\|_{\mathcal{H}} = 1\ \forall x \in \mathbb{R}$, such that $W_1(x) = h_x\ \forall x \in \mathbb{R}$. Explicitly, if $\phi \in L^2(\mathbb{R}, \mathrm{d}x)$,

$$\begin{aligned}
(E(\omega)\phi)(y) &= \int_{\mathbb{R}} \overline{\mathcal{F}}_{\mathbb{R}}(\omega)(x)\langle h_y, h_{y-x}\rangle \phi(y-x)\,\mathrm{d}x \\
&\quad \text{(by unitarity of } \overline{\mathcal{F}}_{\mathbb{R}}) \\
&= \int_{\mathbb{R}} \omega(x)\,\overline{\mathcal{F}}_{\mathbb{R}}\left[\langle h_y, h_{y-\cdot}\rangle \phi(y-\cdot)\right](x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \omega(x)\, e^{iyx}\left\langle h_y, \overline{\mathcal{F}}_{\mathbb{R}}\left[\phi(\cdot)\, h_{\cdot}\right](-x)\right\rangle \mathrm{d}x. \qquad (2.18)
\end{aligned}$$

The corresponding translation covariant observable is thus $E'(\omega) = \overline{\mathcal{F}}^*_{\mathbb{R}} E(\omega)\overline{\mathcal{F}}_{\mathbb{R}}$ for all $\omega \in C_c(\mathbb{R})$.

If $X \in \mathcal{B}(\mathbb{R})$ has finite Lebesgue measure, one can explicitly write down the action of $E(X)$ on a function $\phi \in L^2(\mathbb{R}, \mathrm{d}x)$. If $\psi \in L^1 \cap L^2$, with the notations of Remark 1.2.4 we have

$$\begin{aligned}
\int_{\mathbb{R}} \omega(x)\,\mu_{\phi,\psi}(x) &= \langle \phi, E(\omega)\psi\rangle \\
&= \int_{\mathbb{R}} \mathrm{d}x\omega(x) \int_{\mathbb{R}} e^{iyx}\left\langle \phi(y)\, h_y, \overline{\mathcal{F}}_{\mathbb{R}}\left[\psi(\cdot)\, h_{\cdot}\right](-x)\right\rangle \mathrm{d}y.
\end{aligned}$$

It follows that the complex measure $\mu_{\phi,\psi}$ has density

$$x \longmapsto \int_{\mathbb{R}} e^{iyx}\left\langle \phi(y)\, h_y, \overline{\mathcal{F}}_{\mathbb{R}}\left[\psi(\cdot)\, h_{\cdot}\right](-x)\right\rangle \mathrm{d}y$$

with respect to the Lebesgue measure. We then have

$$\begin{aligned} \langle \phi, E(X)\psi\rangle &= \int_{\mathbb{R}} \mathrm{d}x \chi_X(x) \int_{\mathbb{R}} e^{iyx} \left\langle \phi(y) h_y, \overline{\mathcal{F}}_{\mathbb{R}} [\psi(\cdot) h_\cdot](-x)\right\rangle \mathrm{d}y \\ &= \int_{\mathbb{R}} \mathrm{d}y \overline{\phi(y)} \int_{\mathbb{R}} \chi_X(x) e^{iyx} \left\langle h_y, \overline{\mathcal{F}}_{\mathbb{R}} [\psi(\cdot) h_\cdot](-x)\right\rangle \mathrm{d}x, \end{aligned}$$

where we used $\chi_X \in L^2(\mathbb{R}, \mathrm{d}x)$ to change the order of integration. By density of $L^1 \cap L^2$ it follows

$$(E(X)\psi)(y) = \int_X e^{iyx} \left\langle h_y, \overline{\mathcal{F}}_{\mathbb{R}} [\psi(\cdot) h_\cdot](-x)\right\rangle \mathrm{d}x.$$

If $h_x = h$ is constant for almost all x, eq. (2.18) gives

$$(E(\omega)\phi)(y) = \left[\overline{\mathcal{F}}_{\mathbb{R}}(\omega) * \phi\right](y) = \overline{\mathcal{F}}_{\mathbb{R}} \left[\omega \overline{\mathcal{F}}_{\mathbb{R}}^{-1}(\phi)\right](y),$$

and so the corresponding translation covariant observable is

$$\left(E'(\omega)\phi\right)(y) = \omega(y)\phi(y),$$

i.e. E' is the spectral map associated to the canonical position operator Q.

In [41], the results summarised in this subsection were obtained with a different method.

2.5.2 Covariant phase observables

We give a complete characterisation of the covariance systems based on the one dimensional torus

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \left\{e^{i\theta} \mid \theta \in [0, 2\pi]\right\}.$$

We have

$$\begin{gathered} G = \mathbb{T}, \quad H = \{1\}, \quad G/H = \mathbb{T}, \\ \widehat{G} = H^{\perp} = \{(\mathbb{T} \ni z \longmapsto z^n \in \mathbb{C}) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}, \\ \widehat{G}/H^{\perp} = \{1\}. \end{gathered}$$

We choose $\mu_{G/H} = \frac{1}{2\pi}\mathrm{d}\theta =: \mu_{\mathbb{T}}$, so that $\mu_{H^{\perp}}$ is the counting measure $\mu_{\mathbb{Z}}$ on $\mathbb{Z}$.

Let U be a representation of $\mathbb{T}$. Since $\mathbb{T}$ is compact, we can always assume that U acts diagonally on

$$\mathcal{H} = \bigoplus_{k \in I} F_k,$$

where $I \subset \mathbb{Z}$, and F_k are Hilbert spaces such that $\dim F_k$ is the multiplicity of the representation $k \in \mathbb{Z}$ in U. Explicitly,

$$(U(z)\,\phi_k) = z^k \phi_k$$

for all $z \in \mathbb{T}$ and $\phi_k \in F_k$.

In order to use eq. (2.9), we notice that $F_k = L^2(\mathbb{Z}, \delta_k; F_k)$ (where δ_k is the Dirac measure at k), so that $\rho_k = \delta_k$, . By Corollary 2.4.5, one has that U admits covariant POM's based on $\mathbb{T}$ and that $\alpha_k(j) = \delta_{k,j}$ (where $\delta_{k,j}$ is the Kronecker delta).

Choose an infinite dimensional Hilbert space $\mathcal{M}$ and, for each $k \in I$, fix an isometry W_k from F_k to $\mathcal{M}$. The corresponding covariant POM is given by

$$\begin{aligned} P_j E(\omega) P_k \phi &= \overline{\mathcal{F}}_{\mathbb{T}}(\omega)(j-k)\, W_j^* W_k P_k \phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta}) e^{i(j-k)\theta}\, W_j^* W_k P_k \phi \,\mathrm{d}\theta, \end{aligned}$$

where $\phi \in \mathcal{H}$ and $\omega \in \mathcal{C}(\mathbb{T})$.

Reasoning as in 2.5.1, we can find the explicit form of $E(X)$ for every $X \in \mathcal{B}(\mathbb{T})$. We have

$$P_j E(X) P_k \phi = \frac{1}{2\pi} \int_0^{2\pi} \chi_X(e^{i\theta}) e^{i(j-k)\theta}\, W_j^* W_k P_k \phi \,\mathrm{d}\theta.$$

If $I = \mathbb{N}$ and $\dim F_k = 1\ \forall k \in \mathbb{N}$, U is the **number representation** for a quantum harmonic oscillator. Each subspace F_k is the one dimensional eigenspace of the number operator N corresponding to the eigenvalue k. We have $U\left(e^{i\theta}\right) = e^{i\theta N}$. A transformation by $U\left(e^{i\theta}\right)$ corresponds to a phase shift by θ. Identifying $\mathbb{T}$ with the additive group of the real numbers $\operatorname{mod} 2\pi$, the covariane condition on E thus defines the **covariant phase observable** for the quantum harmonic oscillator. For more details, we refer to refs. [11], [18], [40]. We only remark that there do not exist sharp covariant phase observables. In fact, if we assume the contrary, an easy application of Mackey imprimitivity theorem implies that U is equivalent to the regular representation of $\mathbb{T}$. But this is absurd, since U is strictly contained in the regular representation.

2.5.3 Covariant phase difference observables

As in the previous section, let $\mu_{\mathbb{T}}$ be the normalised Haar measure on the one dimensional torus $\mathbb{T}$. We consider the following representation U of the direct product $G = \mathbb{T} \times \mathbb{T}$ acting on the space $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{T}, \mu_{\mathbb{T}} \otimes \mu_{\mathbb{T}})$ as

$$(U(a,b)\, f)(z_1, z_2) = f\left(a z_1, b^{-1} z_2\right) \qquad (z_1, z_2) \in \mathbb{T} \times \mathbb{T}$$

for all $(a,b) \in \mathbb{T} \times \mathbb{T}$.

Let H be the closed subgroup

$$H = \{(a,b) \in \mathbb{T} \times \mathbb{T} \mid b = a\} \cong \mathbb{T}.$$

We classify all the POM's based on G/H and covariant with respect to U. These describe the phase difference observables for two single mode bosonic fields (for a more detailed account about the physical meaning of such observables and for a different approach to the same problem, see ref. [36]).

We have

$$G = \mathbb{T} \times \mathbb{T}, \quad G/H \cong \mathbb{T}, \quad \widehat{G} = \widehat{\mathbb{T}} \times \widehat{\mathbb{T}} \cong \mathbb{Z} \times \mathbb{Z},$$
$$H^{\perp} = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} \mid k = -j\} \cong \mathbb{Z},$$
$$\widehat{G}/H^{\perp} \cong \mathbb{Z}.$$

We fix $\mu_{G/H} = \mu_{\mathbb{T}}$, so that $\mu_{H^{\perp}} = \mu_{\mathbb{Z}}$.

We choose the following orthonormal basis $(e_{i,j})_{i,j\in\mathbb{Z}}$ of $\mathcal{H}$

$$e_{i,j}(z_1, z_2) = z_1^i z_2^{-j} \qquad (z_1, z_2) \in \mathbb{T} \times \mathbb{T},$$

so that

$$U(a,b)\, e_{i,j} = a^i b^j e_{i,j} \qquad \forall (a,b) \in \mathbb{T} \times \mathbb{T}.$$

Let $F_{i,j} = \mathbb{C}e_{i,j}$, then U acts diagonally on $F_{i,j}$ as the character $(i,j) \in \mathbb{Z}\times\mathbb{Z}$. Then, one can choose as decomposition (2.9)

$$\mathcal{H} = \bigoplus_{i,j\in\mathbb{Z}} F_{i,j} \cong \bigoplus_{i,j\in\mathbb{Z}} L^2(\mathbb{Z} \times \mathbb{Z}, \delta_i \otimes \delta_j; F_{i,j})$$

With the notations of Section 2.4, we have $I = \mathbb{Z} \times \mathbb{Z}$ and $\rho_{i,j} = \delta_i \otimes \delta_j$. It follows that $\mathcal{C}_U^q$ is the equivalence class of $\mu_{\mathbb{Z}}$. With the choice $\nu_U = \mu_{\mathbb{Z}}$, it follows that $\widetilde{\nu} = \mu_{\mathbb{Z}} \otimes \mu_{\mathbb{Z}}$. According to Theorem (2.4.3), U admits covariant POM's and $\alpha_{i,j}(n,m) = \delta_{n,i}\delta_{m,j}$.

With the choice $\mathcal{M} = \mathcal{H}$, we select a map $(i,j) \longmapsto W_{i,j}$, where $W_{i,j}$ is an isometry from $F_{i,j}$ to $\mathcal{H}$. Since $F_{i,j}$ are one dimensional, there exists a family of vectors $(h_{i,j})_{i,j\in\mathbb{Z}}$ in $\mathcal{H}$, with $\|h_{i,j}\|_{\mathcal{H}} = 1 \; \forall (i,j) \in \mathbb{Z} \times \mathbb{Z}$, such that

$$W_{i,j} e_{i,j} = h_{i,j} \qquad \forall (i,j) \in \mathbb{Z} \times \mathbb{Z}.$$

The corresponding covariant POM E is given, for every $\phi \in \mathcal{H}$, by

$$\begin{aligned} P_{l,m} E(\omega) P_{i,j} \phi &= \sum_{h\in\mathbb{Z}} \mathcal{F}_{\mathbb{T}}(\omega)(h)\, \delta_{l-h,i} \delta_{m+h,j} \langle h_{l,m}, h_{i,j} \rangle \; \langle e_{i,j}, \phi \rangle\, e_{l,m} \\ &= \delta_{l+m,i+j}\; \mathcal{F}_{\mathbb{T}}(\omega)(j-m) \langle h_{l,m}, h_{i,j} \rangle \; \langle e_{i,j}, \phi \rangle\, e_{l,m}. \end{aligned}$$

In particular, if $l + m = i + j$, we have

$$\begin{aligned} \langle e_{l,m}, E(\omega)\, e_{i,j} \rangle &= \overline{\mathcal{F}}_{\mathbb{T}}(\omega)(j-m) \langle h_{l,m}, h_{i,j} \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \omega(e^{i\theta}) e^{i(j-m)\theta} \langle h_{l,m}, h_{i,j} \rangle \, \mathrm{d}\theta. \end{aligned}$$

If $l + m \neq i + j$, one has

$$\langle e_{l,m}, E(\omega) e_{i,j} \rangle = 0.$$

If $X \in \mathcal{B}(\mathbb{T} \times \mathbb{T})$, reasoning as in 2.5.1 we can obtain the explicit form of $E(X)$ simply substituing ω with the characteristic function χ_X in the two previous equations.

Chapter 3

The case of an irreducible representation

3.1 Introduction

It is well known ([17], [40], [49]) that, given an irreducible square-integrable representation U of a unimodular group G and a trace class, trace one positive operator T, the family of operators

$$E(X) = \int_X U(g)TU(g^{-1})\mathrm{d}\mu_G(g) \quad X \in \mathcal{B}(G) \tag{3.1}$$

defines a positive operator valued measure based on G and covariant with respect to U (μ_G is a Haar measure on G). In this chapter, we prove that also the converse holds. More precisely, a POM E based on G is covariant with respect to U iff E is expressible in the form of eq. (3.1) for some positive trace one operator T. We will extend this result to non-unimodular groups and to POM's based on the quotient space G/H, where H is a compact subgroup. Moreover, we prove that square-integrability of U is not only a sufficient condition, but it is also necessary in order that U admits covariant POM's based on G/H. Finally, we extend this result to the case of U being an irreducible projective unitary representation of G (see Remark 1.2.6).

The result presented here are a rielaboration of refs. [13] and [20].

We start with the case in which U is a unitary representation and $H = \{e\}$. The more general case in which U is projective and H is compact is discussed in §3.4. As usual, we assume that G is a Hausdorff locally compact second countable topological group.

We fix a left Haar measure μ_G on the group G. We denote by Δ the modular function of G.

We recall some basic properties of square integrable representations (see [30, Theorem 2 and 3]). Recall the definition of the (left) regular representation given in Remark 1.2.1.

Proposition 3.1.1 *Let U be an irreducible representation of G in the Hilbert space $\mathcal{H}$. The following facts are equivalent:*

1. *there exists a vector $u \in \mathcal{H}$ such that*

$$0 < \int_G |\langle U(g)\, u, u \rangle|^2 \, d\mu_G(g) < \infty; \tag{3.2}$$

2. *U is a subrepresentation of the regular representation λ of G.*

If either of the above conditions is satisfied, there exists a selfadjoint injective positive operator C with U-invariant domain and dense range such that

$$U(g)\, C = \Delta(g)^{-\frac{1}{2}}\, CU(g) \qquad \forall g \in G, \tag{3.3}$$

and an isometry $\Sigma : \mathcal{H} \otimes \mathcal{H}^* \longrightarrow L^2(G, \mu_G)$ such that

1. *for all $u \in \mathcal{H}$ and $v \in \operatorname{dom} C$*

$$\Sigma(u \otimes v^*)(g) = \langle U(g)\, Cv, u \rangle \qquad g \in G,$$

2. *for all $g \in G$*

$$\Sigma(U(g) \otimes I_{\mathcal{H}^*}) = \lambda(g)\Sigma,$$

3. *the range of Σ is the isotypic[1] space of U in $L^2(G, \mu_G)$.*

If G is unimodular, the operator C is a multiple of the identity.

If eq. (3.2) is satisfied, U is called **square-integrable**. By eq. (3.3), the selfadjoint operator C is unbounded if G is not unimodular. The square of C is called **formal degree** of U, and it is uniquely determined up to a positive factor which depends on the choice of μ_G.

3.2 Characterisation of E in the case U unitary and $H = \{e\}$

We fix an irreducible representation U of G in the Hilbert space $\mathcal{H}$. The following theorem characterizes all the POM on G covariant with respect to U in terms of positive trace one operators on $\mathcal{H}$ [20].

[1] If U and U' are representations of G in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, and U is irreducible, the **isotypic space** of U in $\mathcal{H}'$ is the maximal invariant subspace $\mathcal{K} \subset \mathcal{H}'$ such that $U'|_{\mathcal{K}}$ decomposes into the direct sum of copies of U.

Theorem 3.2.1 *The irreducible representation U admits a covariant POM based on G if and only if U is square-integrable.*

In this case, let C be the square root of the formal degree of U. There exists a one-to-one correspondence between the set of covariant POMs E on G and the set of positive trace one operators T on $\mathcal{H}$. This correspondence associates to each positive trace one operator T the covariant POM E_T given by

$$\langle u, E_T(X) v\rangle = \int_X \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle d\mu_G(g) \tag{3.4}$$

for all $u, v \in \operatorname{dom} C$ and $X \in \mathcal{B}(G)$.

Proof. Let E be a U-covariant POM based on G. According to the generalized imprimitivity theorem there exists a representation σ of the trivial subgroup $H = \{e\}$ in a Hilbert space $\mathcal{K}$ and an isometry $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$ intertwining U with λ^σ such that

$$E(X) = W^* P^\sigma(X) W$$

for all $X \in \mathcal{B}(G)$. Clearly, σ is the trivial representation in $\mathcal{K}$. We then have

$$\mathcal{H}^\sigma = L^2(G, \mu_G) \otimes \mathcal{K}, \qquad \lambda^\sigma = \lambda \otimes I_{\mathcal{K}}.$$

In particular, U is a subrepresentation of λ, hence it is square-integrable.

Due to Prop. 3.1.1, the operator $W' = (\Sigma^* \otimes I_{\mathcal{K}}) W$ is an isometry from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{K}$ such that

$$\begin{aligned} W'U(g) &= (\Sigma^* \otimes I_{\mathcal{K}}) WU(g) \\ &= (\Sigma^* \otimes I_{\mathcal{K}}) (\lambda(g) \otimes I_{\mathcal{K}}) W \\ &= (U(g) \otimes I_{\mathcal{H}^*} \otimes I_{\mathcal{K}}) (\Sigma^* \otimes I_{\mathcal{K}}) W \\ &= U(g) \otimes I_{\mathcal{H}^* \otimes \mathcal{K}} W'. \end{aligned}$$

Since U is irreducible, by a standard result there is a unit vector $B \in \mathcal{H}^* \otimes \mathcal{K}$ such that

$$W'u = u \otimes B \qquad \forall u \in \mathcal{H}.$$

(see for example [34, p. 342, Proposition 14]). Let $(e_i)_{i \geq 1}$ be an orthonormal basis of $\mathcal{H}$ such that $e_i \in \operatorname{dom} C$, then

$$B = \sum e_i^* \otimes k_i,$$

where $k_i \in \mathcal{K}$ and $\sum_i \|k_i\|^2_{\mathcal{K}} = 1$.

If $u \in \operatorname{dom} C$, one has that

$$\begin{aligned} Wu &= (\Sigma \otimes I_{\mathcal{K}})\,(u \otimes B) \\ &= \sum\nolimits_i \Sigma(u \otimes e_i^*) \otimes k_i \\ &= \sum\nolimits_i \langle U\,(\cdot)\,Ce_i, u\rangle \otimes k_i \\ &= \sum\nolimits_i \left\langle e_i, CU\left(\cdot^{-1}\right) u\right\rangle \otimes k_i \\ &= \sum\nolimits_i (e_i^* \otimes k_i)(CU\left(\cdot^{-1}\right) u), \end{aligned}$$

where the series converges in $\mathcal{H}^\sigma$. On the other hand, for all $g \in G$ the series $\sum_i (e_i^* \otimes k_i)(CU\left(g^{-1}\right) u)$ converges to $BCU\left(g^{-1}\right) u$. Here and in the following we identify $\mathcal{H}^* \otimes \mathcal{K}$ with the space of Hilbert-Schmidt operators mapping $\mathcal{H}$ into $\mathcal{K}$. By uniqueness of the limit

$$(Wu)(g) = BCU\left(g^{-1}\right) u \qquad g \in G.$$

If $u, v \in \operatorname{dom} C$, the corresponding covariant POM is given by

$$\begin{aligned} \langle u, E\,(X)\,v\rangle_{\mathcal{H}} &= \langle Wu, P^\sigma\,(X)\,Wv\rangle_{\mathcal{H}^\sigma} \\ &= \int_G \chi_X\,(g)\left\langle BCU\left(g^{-1}\right) u, BCU\left(g^{-1}\right) v\right\rangle_{\mathcal{K}} \mathrm{d}\mu_G\,(g) \\ &= \int_X \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle_{\mathcal{H}} \mathrm{d}\mu_G\,(g)\,, \end{aligned}$$

where

$$T := B^* B$$

is a positive trace class trace one operator on $\mathcal{H}$.

Conversely, assume that U is square-integrable and let T be a positive trace class trace one operator on $\mathcal{H}$. Then

$$B := \sqrt{T}$$

is a (positive) operator belonging to $\mathcal{H}^* \otimes \mathcal{H}$ such that $B^*B = T$ and $\|B\|_{\mathcal{H}^* \otimes \mathcal{H}} = 1$. The operator W defined by

$$Wv := (\Sigma \otimes I_{\mathcal{H}})\,(v \otimes B) \qquad \forall v \in \mathcal{H}$$

is an isometry intertwining U with the induced representation λ^σ, where σ is the trivial representation of $\{e\}$ in $\mathcal{H}$. Define E_T by

$$E_T\,(X) = W^* P^\sigma\,(X)\,W \qquad X \in \mathcal{B}(G).$$

With the same computation as above, one has that

$$\langle v, E_T\,(X)\,u\rangle = \int_X \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle \mathrm{d}\mu_G\,(g)$$

for all $u, v \in \operatorname{dom} C$.

Finally, we show that the correspondence $T \longmapsto E_T$ is injective. Let T_1 and T_2 be positive trace one operators on $\mathcal{H}$, with $E_{T_1} = E_{T_2}$. Set $T = T_1 - T_2$. Since U is strongly continuous, for all $u, v \in \operatorname{dom} C$ the map

$$\begin{aligned} G \ni g \longmapsto & \left\langle CU\left(g^{-1}\right)u, TCU\left(g^{-1}\right)v\right\rangle \\ & = \Delta(g)^{-1}\left\langle U\left(g^{-1}\right)Cu, TU\left(g^{-1}\right)Cv\right\rangle \in \mathbb{C} \end{aligned}$$

is continuous. Since

$$\int_X \left\langle CU\left(g^{-1}\right)u, TCU\left(g^{-1}\right)v\right\rangle \mathrm{d}\mu_G(g) = \left\langle \left[u, E_{T_1}(X) - E_{T_2}(X)\right]v\right\rangle = 0$$

for all $X \in \mathcal{B}(G)$, we have

$$\left\langle CU\left(g^{-1}\right)u, TCU\left(g^{-1}\right)v\right\rangle = 0 \qquad \forall g \in G.$$

In particular,

$$\langle Cu, TCv\rangle = 0,$$

so that, since C has dense range, $T = 0$. ■

Remark 3.2.2 *If G is unimodular, then $C = \lambda I$, with $\lambda > 0$, and one can normalize μ_G so that $\lambda = 1$. Hence,*

$$E_T(X) = \int_X U(g)\,TU\left(g^{-1}\right)d\mu_G(g) \qquad \forall X \in \mathcal{B}(G),$$

the integral being understood in the weak sense.

3.3 An example: the $ax + b$ group

The $ax + b$ group is the semidirect product $G = \mathbb{R} \times' \mathbb{R}_+$, where we regard $\mathbb{R}$ as additive group and $\mathbb{R}_+$ as multiplicative group. The composition law is

$$(b, a)\left(b', a'\right) = \left(b + ab', aa'\right).$$

The group G is nonunimodular with left Haar measure

$$\mathrm{d}\mu_G(b, a) = a^{-2}\mathrm{d}b\mathrm{d}a$$

and modular function

$$\Delta(b, a) = \frac{1}{a}.$$

Let $\mathcal{H} = L^2((0, +\infty), \mathrm{d}x)$ and $(U, \mathcal{H})$ be the representation of G in $\mathcal{H}$ given by

$$\left[U^+(b, a)f\right](x) = a^{\frac{1}{2}}e^{2\pi ibx}f(ax).$$

It is known ([33], [5]) that U^+ is square-integrable, and the square root of its formal degree is

$$(Cf)(x) = \Delta(0,x)^{\frac{1}{2}} f(x) = x^{-\frac{1}{2}} f(x) \qquad x \in (0,+\infty)$$

acting on its natural domain.

By means of Theorem 3.2.1 every POM based on G and covariant with respect to U^+ is described by a positive trace one operator T according to eq. (3.4). Explicitly, let $(e_i)_{i\geq 1}$ be an orthonormal basis of $\mathcal{H}$ such that $Te_i = \lambda_i e_i$, $\lambda_i \geq 0$, for all i. If $u \in L^2((0,+\infty), \mathrm{d}x)$ is such that $x^{-\frac{1}{2}}u \in L^2((0,+\infty), \mathrm{d}x)$, the U^+-covariant POM corresponding to T is given by

$$\begin{aligned}\langle u, E_T(X)u\rangle &= \int_X \langle CU^+(g^{-1})u, TCU^+(g^{-1})u\rangle \,\mathrm{d}\mu_G(g)\\ &= \int_X \sum\nolimits_i \lambda_i \left|\langle e_i, CU^+(g^{-1})u\rangle\right|^2 \mathrm{d}\mu_G(g)\\ &= \sum\nolimits_i \lambda_i \int_X \left|\int_{\mathbb{R}_+} x^{-\frac{1}{2}} a^{-\frac{1}{2}} e^{-\frac{2\pi i b x}{a}} u\left(\frac{x}{a}\right)\overline{e_i(x)}\mathrm{d}x\right|^2 a^{-2}\mathrm{d}b\mathrm{d}a.\end{aligned}$$

3.4 Characterisation of E for projective representations in the case H is compact

We now suppose that H is a compact subgroup of G and U is an irreducible projective representation of G with multiplier m.

We recall the standard construction that allows to extend U to an irreducible unitary representation $\widetilde{U}$ of the central extension G_m of G associated with the multiplier m. For more details about multipliers and central extensions we refer to [52]. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the multiplicative group of the torus. The group G_m is the set $G \times \mathbb{T}$ with the composition law

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 m(g_1, g_2)).$$

The group G_m can always be endowed with a locally compact second countable Hausdorff topology in which it becomes a topological group. With this topology, $T := \{e\} \times \mathbb{T}$ is a closed subgroup in G_m, and G_m/T is canonically isomorphic to G. Moreover, $G \times \{1\}$ is a Borel subset of G_m. The representation $\widetilde{U}$ is defined by

$$\widetilde{U}(g,z) := z^{-1}U(g) \quad \forall z \in \mathbb{T},\, g \in G.$$

It is a strongly continuous unitary representation of G_m, with $\widetilde{U}(g,1) = U(g)$.

Let $\mu_{\mathbb{T}}$ be the Haar measure of $\mathbb{T}$ with $\mu_{\mathbb{T}}(\mathbb{T}) = 1$. The measure μ_{G_m} of G_m, given by

$$\int_{G_m} \varphi(g,z)\,\mathrm{d}\mu_{G_m}(g,z) = \int_G \mathrm{d}\mu_G(g) \int_{\mathbb{T}} \varphi(g,z)\,\mathrm{d}\mu_{\mathbb{T}}(z)$$

for all $\varphi \in C_c(G_m)$, is clearly a left Haar measure of G_m. We have

$$\begin{aligned}\int_{G_m} \left|\left\langle \widetilde{U}(g,z)\, v, u\right\rangle\right|^2 \mathrm{d}\mu_{G_m}(g,z) &= \int_G \mathrm{d}\mu_G(g) \left|\langle U(g)\, v, u\rangle\right|^2 \int_{\mathbb{T}} \mathrm{d}\mu_{\mathbb{T}}(z) \\ &= \int_G \left|\langle U(g)\, v, u\rangle\right|^2 \mathrm{d}\mu_G(g)\,.\end{aligned}$$

It follows that $\widetilde{U}$ is square integrable if and only if there is some vector $u \in \mathcal{H}$ such that

$$\int_G \left|\langle U(g)\, u, u\rangle\right|^2 \mathrm{d}\mu_G(g) < \infty. \tag{3.5}$$

If eq. (3.5) is satisfied, we say that the projective unitary representation U is **square-integrable**.

Moreover, it is easily checked that a POM E on the quotient $G/H = (G_m/T)/(H_m/T) = G_m/H_m$ is covariant with respect to U if and only if it is covariant with respect to $\widetilde{U}$. We note that, since H is compact, the subgroup H_m is compact in G_m.

The above discussion is summarised in the following lemma.

Lemma 3.4.1 *1. A POM based on $G/H = G_m/H_m$ is covariant with respect to the projective unitary representation U of G if and only if it is covariant with respect to the unitary representation $\widetilde{U}$ of G_m.*

2. The representation $\widetilde{U}$ is square integrable if and only if U is square integrable.

The next theorem is the central result of this chapter. It is a reelaboration of [13, Corollary 5]. See also [6, Theorem II.3.2].

Theorem 3.4.2 *Suppose H is a compact subgroup of G. Assume that U is an irreducible projective unitary representation of G. Then U admits a covariant POM based on G/H if and only if U is square-integrable.*

In this case, let C be the square root of the formal degree of $\widetilde{U}$. There exists a one-to-one correspondence between covariant POMs E on G/H onto the set of positive trace one operators T on $\mathcal{H}$ such that

$$TU(h) = U(h)\, T \quad \forall h \in H. \tag{3.6}$$

This correspondence associates to each positive trace one operator T satisfying eq. (3.6) the covariant POM E_T given by

$$\langle u, E_T(X)\, v\rangle = \int_X \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle d\mu_{G/H}(\dot{g}) \tag{3.7}$$

for all $u, v \in \operatorname{dom} C$ and $X \in \mathcal{B}(G/H)$, where $\mu_{G/H}$ is an invariant measure on G/H.

Proof. By Lemma 3.4.1 and compactness of H_m, possibly switching from G to G_m and from U to $\widetilde{U}$, we can assume that U itself is a unitary representation of G (note that eqs. (3.6) and (3.7) are unaffected by this change). Let μ_H be the invariant measure on H with $\mu_H(H) = 1$. Due to the compactness of H, there exists a left G-invariant measure $\mu_{G/H}$ on G/H such that the following measure decomposition holds

$$\int_G f(g)\,\mathrm{d}\mu_G(g) = \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H f(gh)\,\mathrm{d}\mu_H(h). \tag{3.8}$$

for all $f \in L^1(G, \mu_G)$ (see [33], [34], [27]).

Assume that U is square-integrable and let T be as in the statement of the theorem. By means of eq. (3.4) T defines a POM $\widetilde{E}_T$ based on G and covariant with respect to U. For all $X \in \mathcal{B}(G/H)$ let

$$E_T(X) = \widetilde{E}_T(\pi^{-1}(X)).$$

Clearly, E_T is a POM on G/H covariant with respect to U. Moreover, denoting with χ_X the characteristic function of X, if $u, v \in \operatorname{dom} C$,

$$\begin{aligned}
\langle u, E_T(X)v\rangle &= \int_G \chi_X(\pi(g)) \left\langle CU\left(g^{-1}\right)u, TCU\left(g^{-1}\right)v\right\rangle \mathrm{d}\mu_G(g) \\
&\quad \text{(by eq. (3.8))} \\
&= \int_{G/H} \mathrm{d}\mu_{G/H}(\dot{g}) \int_H \chi_X(\pi(gh)) \left\langle CU(gh)^{-1}u, TCU(gh)^{-1}v\right\rangle \mathrm{d}\mu_H(h) \\
&\quad \text{(by eq. (3.6) and since } \Delta|_H = 1) \\
&= \int_{G/H} \chi_X(\pi(g)) \left\langle CU(g)^{-1}u, TCU(g)^{-1}v\right\rangle \mathrm{d}\mu_{G/H}(\dot{g})
\end{aligned}$$

that is, equation (3.7) holds.

Conversely, let E be a POM on G/H which is covariant with respect to U. For all $Y \in \mathcal{B}(G)$, let l_Y be the function on G given by

$$l_Y(g) = \mu_H(g^{-1}Y \cap H) = \int_H \chi_Y(gh)\mathrm{d}\mu_H(h).$$

Clearly, l_Y is a positive measurable function bounded by 1 and, since μ_H is invariant, for all $h \in H$, $l_Y(gh) = l_Y(g)$. It follows that there is a positive measurable bounded function ℓ_Y on G/H such that $l_Y = \ell_Y \circ \pi$.

Define the operator $\widetilde{E}(Y)$ by means of

$$\widetilde{E}(Y) = \int_{G/H} \ell_Y(\dot{g})\mathrm{d}E(\dot{g}),$$

which is well defined since ℓ_Y is bounded.

We claim that $Y \longmapsto \widetilde{E}(Y)$ is a POM on G covariant with respect to U. Clearly, since ℓ_Y is positive, $\widetilde{E}(Y)$ is a positive operator. Recalling that

$\ell_G = 1$, one has $\widetilde{E}(G) = I$. Let now $(Y_i)_{i\geq 1}$ be a disjoint sequence of $\mathcal{B}(G)$ and $Y = \cup_i Y_i$. Given $g \in G$, since $(g^{-1}Y_i \cap H)_{i\geq 1}$ is a disjoint sequence of $\mathcal{B}(H)$ and $g^{-1}Y \cap H = \cup_i(g^{-1}Y_i \cap H)$, then $\ell_Y = \sum_i \ell_{Y_i}$, where the series converges pointwise. Let $u \in \mathcal{H}$, by monotone convergence theorem, one has that

$$\langle u, \widetilde{E}(Y)\, u\rangle = \sum_i \langle u, \widetilde{E}(Y_i)\, u\rangle.$$

Finally, let $g_1 \in G$, then

$$\begin{aligned}
\widetilde{E}(g_1 Y) &= \int_{G/H} \mu_H(g^{-1}g_1 Y \cap H)\mathrm{d}E(\dot{g}) \\
&\quad (\dot{g} \longrightarrow g_1\dot{g}) \\
&= \int_{G/H} \mu_H(g^{-1}Y \cap H) U(g_1)\, \mathrm{d}E(\dot{g}) U(g_1)^* \\
&= U(g_1)\, \widetilde{E}(Y) U(g_1)^*,
\end{aligned}$$

where we used the fact that E is covariant.

By means of Theorem 3.2.1, U is square-integrable and there is a positive trace one operator T such that, for $u, v \in \operatorname{dom} C$,

$$\left\langle u, \widetilde{E}(Y)\, v\right\rangle = \int_Y \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle \mathrm{d}\mu(g). \tag{3.9}$$

We now show that T satisfies equation (3.6). First of all we claim that, given $h \in H$ and $Y \in \mathcal{B}(G)$,

$$\widetilde{E}(Yh) = \widetilde{E}(Y). \tag{3.10}$$

Indeed, since H is compact, μ_H is both left and right invariant, so that

$$\mu_H(g^{-1}Yh \cap H) = \mu_H((g^{-1}Y \cap H)h) = \mu_H(g^{-1}Y \cap H)$$

and, hence, $\ell_Y = \ell_{Yh}$. By definition of $\widetilde{E}(Y)$, equation (3.10) easily follows. Fixed $h \in H$, by means of equation (3.10) and equation (3.9) one has that

$$\begin{aligned}
&\int_Y \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle \mathrm{d}\mu(g) \\
&\quad = \int_{Yh} \left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle \mathrm{d}\mu(g) \\
&\quad (g \longrightarrow gh) \\
&\quad = \int_Y \left\langle CU(gh)^{-1} u, TCU(gh)^{-1} v\right\rangle \mathrm{d}\mu(g),
\end{aligned}$$

where we used the fact that $\Delta|_H = 1$. Since the equality holds for all $Y \in \mathcal{B}(G)$, then, for a.a. $g \in G$,

$$\begin{aligned}
\left\langle CU\left(g^{-1}\right) u, TCU\left(g^{-1}\right) v\right\rangle &= \left\langle CU(gh)^{-1} u, TCU(gh)^{-1} v\right\rangle \\
&= \left\langle CU(g)^{-1} u, U(h)\, TU(h)^{-1}\, CU(g)^{-1} v\right\rangle.
\end{aligned}$$

Since both sides are continuous functions of g, the equality holds everywhere and equation (3.6) follows by density of $\operatorname{ran} C$.

Let now $X \in \mathcal{B}(G/H)$. Since

$$g^{-1}\pi^{-1}(X) \cap H = \begin{cases} H & \text{if } gH \in X \\ \emptyset & \text{if } gH \notin X \end{cases},$$

then $\ell_{\pi^{-1}(X)} = \chi_X$ and $\widetilde{E}\left(\pi^{-1}(X)\right) = E(X)$. Reasoning as in the first part of the proof one has that $E = E_T$.

The injectivity of the map $T \longmapsto E_T$ easily follows from the injectivity of the map $T \longmapsto \widetilde{E}_T$ from the set of positive trace one operators to the set of U-covariant POM based on G. ■

Remark 3.4.3 *Note that by eq. (3.8) the invariant measure $\mu_{G/H}$ in eq. (3.7) depends only on the normalisation of the Haar measure μ_G, hence on the choice of the operator C, which is uniquely determined up to a positive factor.*

Remark 3.4.4 *If G is unimodular and U is a projective square integrable representation, it is known since [40] that every U-covariant POM is given by eq. (3.7) (with $C = I$). But we actually have more: if U is not square integrable, then U does not admit any covariant POM.*

Remark 3.4.5 *Scutaru shows in ref. [49] that there exists a one-to-one correspondence between covariant POM's E based on G/H with the property*

$$\operatorname{tr} E\,(K) < +\infty \tag{3.11}$$

for all compact sets $K \subset G/H$ and positive trace one operators on $\mathcal{H}$. Theorem 3.4.2 shows that if G is unimodular every *covariant POM E based on G/H shares property (3.11).*

Remark 3.4.6 *Suppose that in Theorem 3.4.2 G is unimodular and U is a unitary representation. Then by eq. (3.7)*

$$E_T\,(X) = \int_X U\,(g)\,TU\,(g)^{-1}\,d\mu_{G/H}\,(\dot{g})$$

for all $X \in \mathcal{B}\,(G/H)$ (the integral being understood in the weak sense). In particular, each U-covariant POM based on G/H admits the representation of eq. (1.8) in §1.3, with operator valued density $E\,(\dot{g}) = U\,(g)\,TU\,(g)^{-1}$ and $\nu = \mu_{G/H}$.

Let σ be a representation of H in a Hilbert space $\mathcal{K}$ and $W : \mathcal{H} \longrightarrow \mathcal{H}^\sigma$ be an operator intertwining U with the representation induced from σ. By irreducibility of U, W is a multiple of an isometry. Then, by Theorem

1.3.2, there exists an operator $A : \mathcal{H} \longrightarrow \mathcal{K}$ such that $AU(h) = \sigma(h) A$ for all $h \in H$, and

$$(Wv)(g) = AU(g)^{-1} v \quad \forall v \in \mathcal{H}.$$

Since

$$E(X) := W^* P^\sigma (X) W = \int_X U(g) A^* A U(g)^{-1} \, d\mu_{G/H}(\dot{g})$$

is a multiple of a covariant POM, by Theorem 3.4.2 A^*A is trace class, i.e. A is a Hilbert-Schmidt operator.

Remark 3.4.7 *Suppose that H is a closed subgroup of G such that*

1. *H contains a closed subgroup Z which is central in G;*
2. *H/Z is compact.*

Let U be an irreducible representation of G in the Hilbert space $\mathcal{H}$ and γ be the character of Z such that $U(z) = \gamma(z) I_{\mathcal{H}}$ for all $z \in Z$. Fix a Borel section $s : G/Z \longrightarrow G$, and define

$$\widehat{U}(\dot{g}) := U(s(\dot{g})) \quad \forall \dot{g} \in G/Z.$$

It is easily checked that $\widehat{U}$ is a projective unitary representation of the quotient group G/Z with multiplier

$$m(\dot{g}_1, \dot{g}_2) = \gamma\left(s(\dot{g}_1 \dot{g}_2) s(\dot{g}_2)^{-1} s(\dot{g}_1)^{-1}\right).$$

Moreover, a POM based on $G/H = (G/Z)/(H/Z)$ is U-covariant iff it is $\widehat{U}$-covariant. By Theorem 3.4.2, then U admits covariant POM's iff there is some vector $u \in \mathcal{H}$ such that

$$\int_{G/Z} |\langle U(g) u, u\rangle|^2 \, d\mu_{G/Z}(\dot{g}) < \infty. \tag{3.12}$$

*In this case, it is easily checked that the U-covariant POM's are given again by formula (3.7). An irreducible unitary representation U of G satisfying eq. (3.12) is called **square-integrable modulo** Z.*

3.5 Two examples

3.5.1 The isochronous Galilei group

The following example is taken from [13]. Consider a free nonrelativistic spin-0 particle with mass m in the Euclidean space $\mathbb{R}^3$. Its associated Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}\vec{x})$. The symmetry group of the system is the isochronous Galilei group. We recall that this group is the topological space $G = \mathbb{R}^3 \times \mathbb{V}^3 \times SO(3)$, where $\mathbb{R}^3$ is the 3-dimensional vector group of space translations, $\mathbb{V}^3 \simeq \mathbb{R}^3$ is the 3-dimensional vector group of velocity boosts and $SO(3)$ is the group of rotations (connected with the identity). The composition law of G is

$$(\vec{a}, \vec{v}, R)\left(\vec{a}', \vec{v}', R'\right) = \left(\vec{a} + R\vec{a}', \vec{v} + R\vec{v}', RR'\right).$$

The group G acts in $\mathcal{H}$ through the irreducible projective unitary representation U defined as follows

$$[U(\vec{a}, \vec{v}, R)\phi](\vec{x}) = e^{im\vec{v}\cdot(\vec{x}-\vec{a})}\phi\left(R^{-1}(\vec{x}-\vec{a})\right). \tag{3.13}$$

The phase space of the system is $\Omega = \mathbb{R}^3 \times \mathbb{P}^3$. The action of an element $g = (\vec{a}, \vec{v}, R) \in G$ on a point $x = (\vec{q}, \vec{p}) \in \Omega$ is given by

$$g[x] = (\vec{a} + R\vec{q}, m\vec{v} + R\vec{p}).$$

The stability subgroup at the point $\left(\vec{0}, \vec{0}\right)$ is the compact subgroup $H = SO(3)$. In particular, Ω is isomorphic to G/H by means of

$$(\vec{q}, \vec{p}) \longmapsto \pi\left(\vec{q}, \frac{\vec{p}}{m}, I\right). \tag{3.14}$$

A **covariant phase space observable** is a POM E based on Ω and covariant with respect to U. To apply Theorem 3.4.2 and classify such POM's we need to check the square-integrability of U. Denoting by $\mathrm{d}R$ the normalised Haar measure of $SO(3)$, we fix in G the Haar measure

$$\mathrm{d}\mu_G(\vec{a}, \vec{v}, R) = \frac{m}{(2\pi)^3}\mathrm{d}\vec{a}\mathrm{d}\vec{v}\mathrm{d}R.$$

If $\psi \in L^2\left(\mathbb{R}^3, \mathrm{d}\vec{x}\right)$, we have

$$\begin{aligned}
&\int_G |\langle U(\vec{a}, \vec{v}, R)\psi, \psi\rangle|^2 \mathrm{d}\mu_G(\vec{a}, \vec{v}, R) = \\
&\quad = \int_{\mathbb{R}^3\times\mathbb{V}^3\times SO(3)} \left|\int_{\mathbb{R}^3} \psi(\vec{x})\, e^{-im\vec{v}\cdot(\vec{x}-\vec{a})}\overline{\psi\left(R^{-1}(\vec{x}-\vec{a})\right)}\mathrm{d}\vec{x}\right|^2 \frac{m\mathrm{d}\vec{a}\mathrm{d}\vec{v}\mathrm{d}R}{(2\pi)^3} \\
&\quad = \int_{\mathbb{R}^3\times SO(3)} \left[\int_{\mathbb{V}^3} \left|\mathcal{F}\left(\psi(\cdot)\,\overline{\psi\left(R^{-1}(\cdot-\vec{a})\right)}\right)(m\vec{v})\right|^2 m\mathrm{d}\vec{v}\right] \mathrm{d}\vec{a}\mathrm{d}R \\
&\quad = \int_{\mathbb{R}^3\times SO(3)} \left[\int_{\mathbb{V}^3} \left|\psi(\vec{x})\,\overline{\psi\left(R^{-1}(\vec{x}-\vec{a})\right)}\right|^2 \mathrm{d}\vec{x}\right] \mathrm{d}\vec{a}\mathrm{d}R = \|\psi\|^4.
\end{aligned}$$

Thus, U is a square-integrable representation with formal degree $C^2 = I$. Choosing $\mathrm{d}\mu_{G/H}(\vec{a}, \vec{v}) = \frac{m}{(2\pi)^3}\mathrm{d}\vec{a}\mathrm{d}\vec{v}$ and recalling identification (3.14), every U-covariant POM based on Ω has the form

$$E_T(X) = \frac{1}{(2\pi)^3}\int_X U_{\left(\vec{q}, \frac{\vec{p}}{m}, I\right)} T U^*_{\left(\vec{q}, \frac{\vec{p}}{m}, I\right)} \mathrm{d}\vec{q}\mathrm{d}\vec{p} \tag{3.15}$$

for all $X \in \mathcal{B}(\Omega)$, where T is a positive trace one operator commuting with $U|_{SO(3)}$.

We now characterize the positive trace one operators T commuting with $U|_{SO(3)}$. We have the factorization

$$L^2\left(\mathbb{R}^3, \mathrm{d}\vec{x}\right) = L^2\left(S^2, \mathrm{d}\Omega\right) \otimes L^2\left(\mathbb{R}_+, r^2 \mathrm{d}r\right).$$

Denoting with l the representation of $SO(3)$ acting in $L^2\left(S^2, \mathrm{d}\Omega\right)$ by left translations, we have

$$U|_{SO(3)} = l \otimes I.$$

The representation $\left(l, L^2\left(S^2, \mathrm{d}\Omega\right)\right)$ decomposes into

$$L^2\left(S^2, \mathrm{d}\Omega\right) = \bigoplus_{\ell \geq 0} M_\ell,$$

where each irreducible inequivalent subspace M_ℓ is generated by the spherical harmonics $(Y_{\ell m})_{-\ell \leq m \leq \ell}$. We have

$$L^2\left(\mathbb{R}^3, \mathrm{d}\vec{x}\right) = \left(\bigoplus_{\ell \geq 0} M_\ell\right) \otimes L^2\left(\mathbb{R}_+, r^2 \mathrm{d}r\right) = \bigoplus_{\ell \geq 0} \left(M_\ell \otimes L^2\left(\mathbb{R}_+, r^2 \mathrm{d}r\right)\right).$$

Let $P_\ell : L^2\left(S^2, \mathrm{d}\Omega\right) \longrightarrow L^2\left(S^2, \mathrm{d}\Omega\right)$ be the orthogonal projection onto the subspace M_ℓ. If T commutes with $l \otimes I$, one has

$$T\left(P_\ell \otimes I\right) = \left(P_\ell \otimes I\right) T,$$

where $P_\ell \otimes I$ projects onto $M_\ell \otimes L^2\left(\mathbb{R}_+, r^2 \mathrm{d}r\right)$. Given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and an irreducible representation π acting in a Hilbert space $\mathcal{K}$, a standard result asserts that the operators interwining $\pi \otimes I_{\mathcal{H}_1}$ and $\pi \otimes I_{\mathcal{H}_2}$ are exactly the tensor product $I_{\mathcal{K}} \otimes \mathcal{L}\left(\mathcal{H}_1, \mathcal{H}_2\right)$ (see [34, p. 342, Proposition 14]). Since M_ℓ is irreducible, this implies

$$T\left(P_\ell \otimes I\right) = P_\ell \otimes T_\ell$$

with $T_\ell \in \mathcal{L}\left(L^2\left(\mathbb{R}_+, r^2 \mathrm{d}r\right)\right)$. We then have

$$T = \sum_\ell T\left(P_\ell \otimes I\right) = \sum_\ell P_\ell \otimes T_\ell.$$

Since T is a positive trace one operator, each T_ℓ is positive and

$$1 \equiv \sum_\ell \dim M_\ell \operatorname{tr} T_\ell = \sum_\ell (2\ell + 1) \operatorname{tr} T_\ell. \tag{3.16}$$

It follows that the operators T associated to the U-covariant POM's E by means of equation (3.15) are all the operators of the form

$$T = \sum_\ell P_\ell \otimes T_\ell$$

with T_ℓ positive trace class operators satisfying equation (3.16).

3.5.2 Covariant phase space observables in one dimension.

In this subsection, the quantum system $\mathcal{S}$ is a nonrelativistic spin-0 particle with mass m in the one dimensional space $\mathbb{R}$. The Hilbert space of $\mathcal{S}$ is $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ ($\mathrm{d}x$ being the Lebesgue measure of $\mathbb{R}$). The symmetry group of the system is the isochronous Galilei group in one dimension. This is the additive abelian group $G = \mathbb{R} \times \mathbb{V}$ ($\mathbb{V} \simeq \mathbb{R}$), which acts on $\mathcal{H}$ by means of the projective unitary representation U given by

$$[U(a,v)\psi](x) = e^{imv(x-a)}\psi(x-q) \quad \forall \psi \in \mathcal{H}$$

(compare with eq. (3.13)). The one dimensional phase space is $\Omega = \mathbb{R} \times \mathbb{P}$, on which G acts by

$$(a,v)[(q,p)] = (q+a, p+mv).$$

A **covariant phase space observable** is thus a POM based on Ω and covariant with respect to U.

To simplify our notations in the next chapters, we introduce the group $G' = \mathbb{R} \times \mathbb{P}$ of space translations and momentum boosts, also called the group of phase space translations (see [11], [40]). We define the projective unitary representation of G' on $\mathcal{H}$, given by

$$[W(q,p)\psi](x) = e^{ip\left(x-\frac{q}{2}\right)}\psi(x-q) \quad \forall \psi \in \mathcal{H}. \tag{3.17}$$

This can be written

$$W(q,p) = e^{i(pQ-qP)},$$

where

$$[Q\psi](x) = x\psi(x), \qquad [P\psi](x) = -i\frac{\mathrm{d}}{\mathrm{d}t}\psi(x)$$

defined on their natural domains are the usual selfadjoint generators of boosts and translations. The group G' acts on the phase space Ω by translations:

$$(q,p)\left[\left(q',p'\right)\right] = \left(q+q', p+p'\right).$$

It is easily seen that a POM $E : \mathcal{B}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ is a covariant phase space observable iff it is covariant with respect to the representation W of G'. In the following two chapters, we will work with G' and W rather than G and U.

We endow G' with the Haar measure

$$\mathrm{d}\mu_{G'}(p,q) = \frac{1}{2\pi}\mathrm{d}p\mathrm{d}q.$$

It is known ([17], [33], [40]) that the representation W is square-integrable, with $C = 1$. It follows from Theorem 3.2.1 that *any* W-covariant POM E based on Ω is of the form

$$E(X) = \frac{1}{2\pi}\int_X e^{i(pQ-qP)}Te^{-i(pQ-qP)}\mathrm{d}p\mathrm{d}q \qquad X \in \mathcal{B}(\mathbb{R} \times \mathbb{P})$$

for some positive trace one operator T on $\mathcal{H}$. This result was known since [40] and [54] (although in the book of Holevo it follows from a more general result, while the proof of Werner is specific for the case of the Heisenberg group).

Chapter 4

Covariant position and momentum observables

4.1 Introduction

In this chapter, we define the position and momentum observables for a nonrelativistic quantum particle by means of their covariance property under the transformations of the isochronous Galilei group.

We will show that these observables are a smeared or 'fuzzy' version of the canonical sharp position and momentum observables defined in eqs. (4.1) and (4.2) below (for a detailed account about fuzzy observables, we refer to [11] and [35]).

Moreover, we will characterise some operational properties of the position and momentum observables, such as regularity and state distinction power. We also introduce a variant of the concept of regularity which has a quite transparent meaning for these observables. We call it *α-regularity*. α-regularity allows one to characterise the *limit of resolution* of a position or a momentum observable (for more details, we refer to §4.7)

Most of our results will be given only in the one dimensional case, since their extension to the particle in the Euclidean space is quite straightforward. Moreover, we will work with the group of space translations and momentum boosts of §3.5.2 (and with its three dimensional analogue) rather than with the Galilei group, since this will slightly simplify our notations (essentially, this drops the mass factor in our formulas).

Finally, in the next chapter we will conclude our discussion on position and momentum observables treating the problem of their coexistence and joint measurability.

Here we stress that, in different approaches, position and momentum observables can be defined in different ways, sometimes even without referring to their properties of covariance. We shall not enter into details about this. For more information, see [11], [55] and references therein.

If not explicitely stated otherwise, the results of this chapter are all taken from [14].

We end this section introducing some notations that will be used in the following. Since we will always be concerned with $\mathbb{R}^n$ endowed with the Lebesgue measure $\mathrm{d}x^n$, we will use the abbreviated notation $L^p(\mathbb{R}^n)$ for $L^p(\mathbb{R}^n, \mathrm{d}x^n)$. The Fourier transform of any $f \in L^1(\mathbb{R}^n)$ is denoted by $\hat{f}$. We set also $\hat{f} = \mathcal{F}(f)$ to denote the Fourier-Plancherel transform of any $f \in L^2(\mathbb{R}^n)$, and similarly $\hat{\mu} = \mathcal{F}(\mu)$ is the Fourier-Stieltjes transform of any complex Borel measure μ on $\mathbb{R}^n$.

If $E : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ is a POM based on the measurable space $(\Omega, \mathcal{A}(\Omega))$, an element $E(X) \in \operatorname{ran} E \subset \mathcal{L}(\mathcal{H})$ is called an **effect**. The effects O and I are called **trivial**.

4.2 Definition of the observables of position and momentum on $\mathbb{R}$

Let us consider a nonrelativistic spin-0 particle living in the one-dimensional space $\mathbb{R}$ and fix $\mathcal{H} = L^2(\mathbb{R})$. Let U and V be the one-parameter unitary representations on $\mathcal{H}$ related to the groups of space translations and momentum boosts. They act on $\varphi \in \mathcal{H}$ as

$$\begin{aligned} [U(q)\varphi](x) &= \varphi(x-q), \\ [V(p)\varphi](x) &= e^{ipx}\varphi(x). \end{aligned}$$

Let P and Q be the selfadjoint operators generating U and V, that is, $U(q) = e^{-iqP}$ and $V(p) = e^{ipQ}$ for every $q, p \in \mathbb{R}$. We denote by Π_P and Π_Q the spectral decompositions of the operators P and Q, respectively. They have the form

$$[\Pi_Q(X)\varphi](x) = \chi_X(x)\varphi(x), \tag{4.1}$$

$$\Pi_P(X) = \mathcal{F}^{-1}\Pi_Q(X)\mathcal{F}. \tag{4.2}$$

The projection valued measure $\Pi_Q : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ has the property that, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,

$$U(q)\Pi_Q(X)U(q)^* = \Pi_Q(X+q), \tag{4.3}$$

$$V(p)\Pi_Q(X)V(p)^* = \Pi_Q(X). \tag{4.4}$$

More generally, the abelian group $G = \mathbb{R} \times \mathbb{P}$ of the space translations and momentum boosts acts on the one dimensional space $\mathbb{R}$ by

$$(q,p)[x] = x + q \quad \forall x \in \mathbb{R},\ (q,p) \in \mathbb{R} \times \mathbb{P}.$$

On the other hand, its action on the Hilbert space $L^2(\mathbb{R})$ of the quantum particle is given by the projective unitary representation W defined in eq. (3.17) (see §3.5.2). So, a position observable $E : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ must satisfy the covariance condition

$$W(q,p)\,E(X)\,W(q,p)^* = E(X+q) \quad \forall X \in \mathcal{B}(\mathbb{R}).$$

Since $W(q,p) = e^{iqp/2} U(q)\, V(p)$, it is easy to check that the last equation is equivalent to the analogues of eqs. (4.3) and (4.4) with the sharp observable Π_Q replaced by the POM E. We thus take covariance under translations and invariance under momentum boosts as the defining properties of a general position observable.

Definition 4.2.1 *An observable $E : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ is a* ***position observable on*** $\mathbb{R}$ *if, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,*

$$\begin{aligned} U(q)E(X)U(q)^* &= E(X+q), && (4.5)\\ V(p)E(X)V(p)^* &= E(X). && (4.6) \end{aligned}$$

We will denote by $\mathcal{POS}_{\mathbb{R}}$ the convex set of all position observables on $\mathbb{R}$.

The projection valued position observable Π_Q is called the **canonical (sharp) position observable**.

In an analogous way we define a momentum observable to be an observable which is covariant under momentum boosts and invariant under translations.

Definition 4.2.2 *An observable $F : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ is a* ***momentum observable on*** $\mathbb{R}$ *if, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,*

$$\begin{aligned} U(q)F(X)U(q)^* &= F(X), && (4.7)\\ V(p)F(X)V(p)^* &= F(X+p). && (4.8) \end{aligned}$$

Since $\mathcal{F}U(q) = V(-q)\mathcal{F}$ and $\mathcal{F}V(p) = U(p)\mathcal{F}$, the sharp observable $\Pi_P = \mathcal{F}^{-1}\Pi_Q\mathcal{F}$ satisfies (4.7) and (4.8). It is called the **canonical (sharp) momentum observable**. Moreover, an observable E is a position observable if and only if $\mathcal{F}^{-1}E\mathcal{F}$ is a momentum observable. Therefore, in the following we will restrict ourselves to the study of position observables, the results of Sections 4.3, 4.6 and 4.7 being easily converted to the case of momentum observables.

Remark 4.2.3 *We classified in §2.5.1 the POM's $E : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ which fulfill only eq. (4.5). In some articles the name 'position observables' is used to denote these observables, without requiring the additional property of invariance under boosts. In §2.5.1 we say that such observables are 'translation covariant observables', and we reserve the name 'position observables' only to the observables satisfying eqs. (4.5) and (4.6). In §4.3 it is shown,* *in particular, that every position observable is commutative. However, using the classification of translation covariant observables given in section 2.5.1, it is easy to check that there exist noncommutative localization observables. Thus, the position observables are a strict subset of the set of the translation covariant observables.*

4.3 The structure of position observables

Let $\rho : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a probability measure. For any $X \in \mathcal{B}(\mathbb{R})$, the map $q \mapsto \rho(X-q)$ is bounded and measurable, and hence the equation

$$E_\rho(X) = \int \rho(X-q) \, \mathrm{d}\Pi_Q(q) \tag{4.9}$$

defines a bounded positive operator. The map

$$\mathcal{B}(\mathbb{R}) \ni X \mapsto E_\rho(X) \in \mathcal{L}(\mathcal{H})$$

is an observable. It is straightforward to verify that the observable E_ρ satisfies the covariance condition (4.5) and the invariance condition (4.6), hence it is a position observable on $\mathbb{R}$. Denote by δ_t the Dirac measure concentrated at t. The observable E_{δ_0} is the canonical position observable Π_Q. We may also write

$$\Pi_Q(X) = \int \delta_0(X-q) \, \mathrm{d}\Pi_Q(q) \tag{4.10}$$

and comparing (4.9) to (4.10) we note that E_ρ is obtained when the sharply concentrated Dirac measure δ_0 is replaced by the probability measure ρ. The observable E_ρ admits an interpretation as an imprecise, or fuzzy, version of the canonical position observable Π_Q, unsharpness being characterised by the probability measure ρ (see [1], [2], [3], [35] for further details).

The following is the central result of this chapter.

Proposition 4.3.1 *Any position observable E on $\mathbb{R}$ is of the form $E = E_\rho$ for some probability measure $\rho : \mathcal{B}(\mathbb{R}) \to [0,1]$.*

Since the proof of Proposition 4.3.1 is quite long and can be given in a slightly greater generality which will be useful in the following, we postpone it to the next section.

We denote by $M(\mathbb{R})$ the set of complex measures on $\mathbb{R}$. $M_1^+(\mathbb{R})$ is the convex set of probability measures. Proposition 4.3.1 thus estabilishes a map $\rho \longmapsto E_\rho$ from $M_1^+(\mathbb{R})$ onto the convex set $\mathcal{POS}_\mathbb{R}$ of the position observables on $\mathbb{R}$. It is immediately checked that this map is convex. The next proposition (proved in [15]) shows that it is an isomorphism of convex sets.

Proposition 4.3.2 *Let $\rho_1, \rho_2 \in M_1^+(\mathbb{R})$, $\rho_1 \neq \rho_2$. Then $E_{\rho_1} \neq E_{\rho_2}$.*

Proof. For $\psi \in \mathcal{H}$, we define the real measure λ_ψ by

$$\lambda_\psi (X) = \langle \psi, (E_{\rho_1} (X) - E_{\rho_2} (X)) \psi \rangle = \mu_\psi * (\rho_1 - \rho_2) (X) ,$$

where $*$ is the convolution and $\mathrm{d}\mu_\psi (x) = |\psi (x)|^2 \mathrm{d}x$. Taking the Fourier-Stieltjes transform we get

$$\hat{\lambda}_\psi = \hat{\mu}_\psi \cdot (\hat{\rho}_1 - \hat{\rho}_2) ,$$

where $\hat{\lambda}_\psi$, $\hat{\mu}_\psi$, $\hat{\rho}_1$ and $\hat{\rho}_2$ are continuous functions. By injectivity of the Fourier-Stieltjes transform we have $\hat{\rho}_1 \neq \hat{\rho}_2$. Thus, choosing ψ such that $\widehat{|\psi|^2}(p) \neq 0$ for every $p \in \mathbb{R}$, we have $\hat{\lambda}_\psi \neq 0$. This means that $\lambda_\psi \neq 0$ and hence, $E_{\rho_1} \neq E_{\rho_2}$. ■

As we will see in Proposition 4.5.2 below, a position observable E_ρ is a sharp observable if and only if $\rho = \delta_x$ for some $x \in \mathbb{R}$, where δ_x is the Dirac measure concentrated at x. Since the Dirac measures are the extreme elements of the convex set $M_1^+(\mathbb{R})$, from the above discussion the following fact follows [15].

Proposition 4.3.3 *The sharp position observables are the extreme elements of the convex set $\mathcal{POS}_\mathbb{R}$.*

The following useful property of a position observable is proved in [37]. Here we give a slightly modified proof.

Proposition 4.3.4 *If $E : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ satisfies eq. (4.5), then $E(X) = O$ if and only if the Borel set X has zero Lebesgue measure.*

Proof. This is exactly as in the first part of the proof of Lemma 1.3.1. Let $(v_n)_{n\geq 1}$ be a countable dense subset in $\mathcal{H}$, and let μ be the bounded measure on $\mathbb{R}$ given by

$$\mu (X) = \sum\nolimits_n 2^{-n} \|v_n\|^{-2} \langle v_n, E (X) v_n \rangle .$$

We have $\mu (X) = 0$ iff $E (X) = 0$. By this fact and the translational covariance of E we have $\mu (X + x) = 0$ iff $\mu (X) = 0$, i.e. μ is a quasi invariant measure on $\mathbb{R}$. So, μ is equivalent to the Lebesgue measure, thus proving our claim. ■

We conclude this section with a standard example.

Example 4.3.5 *Let $\rho \in M_1^+(\mathbb{R})$ be absolutely continuous with respect to the Lebesgue measure and let $e \in L^1(\mathbb{R})$ be the corresponding Radon-Nikodým derivative. Then (4.9) can be written in the form*

$$E_\rho(X) = (\chi_X * \tilde{e}) (Q) , \tag{4.11}$$

where $\tilde{e}(q) = e(-q)$ and $*$ denotes the convolution.

4.4 Translation covariant and boost invariant observables in dimension n

Let $N = \mathbb{R}^{n+1}$ and $H = \mathbb{R}^n$, with the usual structure of additive abelian groups. Denote with (p,t), $p \in \mathbb{R}^n$, $t \in \mathbb{R}$, an element of N. Let H act on N as

$$\alpha_q(p,t) = (p, t + q \cdot p) \quad q \in H,\ (p,t) \in N.$$

The Heisenberg group[1] is the semidirect product $G = N \times_\alpha H$. We recall that such a group is the topological set $G = N \times H$ endowed with the composition law

$$((p,t),q)((p',t'),q') = ((p+p', t+t'+q \cdot p'), q+q'). \tag{4.12}$$

Let W be the following irreducible unitary representation of G acting in $L^2(\mathbb{R}^n)$

$$[W((p,t),q)f](x) = e^{-i(t-p\cdot x)} f(x-q).$$

Clearly, $W((0,0),q) = U(q)$, $W((p,0),0) = V(p)$, and $W((0,t),0) = e^{-it}$. The groups H and G/N are naturally identified. With such an identification, the canonical projection $\pi : G \longrightarrow G/N$ is

$$\pi((p,t),q) = q,$$

and an element $((p,t),q) \in G$ acts on $q_0 \in H$ as

$$((p,t),q)[q_0] = \pi(((p,t),q)((0,0),q_0)) = q + q_0.$$

A POM E based on $\mathbb{R}^n$ and acting in $L^2(\mathbb{R}^n)$ satisfies the analogues of eqs. (4.5), (4.6) in dimension n if, and only if, for all $X \in \mathcal{B}(\mathbb{R}^n)$ and $((p,t),q) \in G$,

$$W((p,t),q)E(X)W((p,t),q)^* = E(X+q), \tag{4.13}$$

i.e. if and only if E is a W-covariant POM based on G/N. By virtue of the generalized imprimitivity theorem of §1.2, E is W-covariant if and only if there exists a representation σ of N and an isometry L intertwining W with the induced representation $\mathrm{ind}_N^G(\sigma)$ such that

$$E(X) = L^* P^\sigma(X) L$$

[1]Usually, the Heisenberg group is defined as the topological set $\mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n$ with composition law

$$(t,q,p)(t',q',p') = \left(t + t' + \frac{1}{2}(q \cdot p' - p \cdot q'), q + q', p + p'\right).$$

It is easy to show that this is isomorphic to our definition in eq. (4.12).

for all $X \in \mathcal{B}(\mathbb{R}^n)$. Since $\mathrm{ind}_N^G(\sigma) \subset \mathrm{ind}_N^G(\sigma')$ (as imprimitivity systems) if $\sigma \subset \sigma'$ (as representations), it is not restrictive to assume that such a σ has constant infinite multiplicity, so that there exists a positive Borel measure μ_σ on $\widehat{N} = \mathbb{R}^{n+1}$ and an infinite dimensional Hilbert space $\mathcal{H}$ such that σ is the diagonal representation acting in $L^2\left(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}\right)$, i.e.

$$[\sigma(p,t)\,\phi](h,k) = e^{ih\cdot p}e^{ikt}\phi(h,k)\,.$$

Denote with $\gamma_{h,k}$, $h \in \mathbb{R}^n$, $k \in \mathbb{R}$ the following character of N

$$\gamma_{h,k}(p,t) = e^{ih\cdot p}e^{ikt}.$$

The action of H on $\widehat{N}$ is given by

$$(q\cdot\gamma_{h,k})(p,t) = \gamma_{h,k}(\alpha_{-q}(p,t)) = e^{i(h-kq)\cdot p}e^{ikt},$$

or in other words

$$q\cdot\gamma_{h,k} = \gamma_{h-kq,k}.$$

If $k \neq 0$, the H-orbit passing through $\gamma_{h,k}$ is thus

$$\mathcal{O}_{\gamma_{h,k}} = \mathbb{R}^n \times \{k\}$$

and the corresponding stability subgroup is

$$H_{\gamma_{h,k}} = \{0\}.$$

From Mackey's theory ([33], [52]) it follows that the representations

$$\rho_{h,k} := \mathrm{ind}_N^G(\gamma_{h,k})$$

are irreducible if $k \neq 0$, $\rho_{h,k}$ and $\rho_{h',k'}$ are inequivalent if $k \neq k'$ and, fixed $k \neq 0$, $\rho_{h,k}$ and $\rho_{h',k}$ are equivalent.

The representation $\rho := \mathrm{ind}_N^G(\sigma)$ acts on $L^2\left(\mathbb{R}^n, \mathrm{d}x; L^2\left(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}\right)\right)$ according to

$$[\rho((p,t),q)\,f](x) = \sigma(p, t - p\cdot x)\,f(x-q)$$

(here we are using the second realisation of $\mathrm{ind}_N^G(\sigma)$ which we described in §1.2). Using the fact that σ acts diagonally in $L^2\left(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}\right)$ and the identification $L^2\left(\mathbb{R}^n, \mathrm{d}x; L^2\left(\mathbb{R}^{n+1}, \mu_\sigma; \mathcal{H}\right)\right) \cong L^2\left(\mathbb{R}^n \times \mathbb{R}^{n+1}, \mathrm{d}x \otimes \mathrm{d}\mu_\sigma(x); \mathcal{H}\right)$, we find that ρ acts on $L^2\left(\mathbb{R}^n \times \mathbb{R}^{n+1}, \mathrm{d}x \otimes \mathrm{d}\mu_\sigma(x); \mathcal{H}\right)$ as

$$[\rho((p,t),q)\,f](x,h,k) = e^{ih\cdot p}e^{ik(t-p\cdot x)}f(x-q,h,k)\,.$$

Write $\mu_\sigma = \mu_{\sigma_1} + \mu_{\sigma_2}$, where $\mu_{\sigma_1} \perp \mu_{\sigma_2}$ and $\mu_{\sigma_2}\left(\mathcal{O}_{\gamma_{0,-1}}\right) = 0$, and let $\sigma = \sigma_1 \oplus \sigma_2$ be the corresponding decomposition of σ. We then have

$$\mathrm{ind}_N^G(\sigma) = \mathrm{ind}_N^G(\sigma_1) \oplus \mathrm{ind}_N^G(\sigma_2),$$

where the two representations in the sum are disjoint and the sum is a direct sum of imprimitivity systems. So, since $W \simeq \mathrm{ind}_N^G(\gamma_{0,-1})$, it is not restrictive to assume $\sigma = \sigma_1$, or, in other words, that μ_σ is concentrated in the orbit $\mathcal{O}_{\gamma_{0,-1}} = \mathbb{R}^n \times \{-1\} \cong \mathbb{R}^n$.

Let T be the following unitary operator in $L^2(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{d}x \otimes \mathrm{d}\mu_\sigma(x); \mathcal{H})$:

$$[Tf](x,h) = f(x+h,h).$$

If we define the representation $\hat{\rho}$, given by

$$[\hat{\rho}((p,t),q) f](x,h) = e^{-i(t-p\cdot x)} f(x-q,h),$$

then T intertwines $\hat{\rho}$ with ρ. Since $\hat{\rho} \simeq W \otimes I_{L^2(\mathbb{R}^n,\mu_\sigma;\mathcal{H})}$ and W is irreducible, every isometry intertwining W with $\hat{\rho}$ has the form

$$[\widetilde{L}f](x,h) = f(x)\varphi(h) \quad \forall f \in L^2(\mathbb{R}^n)$$

for some $\varphi \in L^2(\mathbb{R}^n, \mu_\sigma; \mathcal{H})$ with $\|\varphi\|_{L^2} = 1$. The most general isometry L intertwining W with ρ has then the form $L = T\widetilde{L}$ for some choice of φ, and the corresponding observable is given by

$$\begin{aligned} \langle g, E(X)f \rangle &= \langle g, L^* P^\sigma(X) L f \rangle = \left\langle T\widetilde{L}g, P^\sigma(X) T\widetilde{L} f \right\rangle \\ &= \int_{\mathbb{R}^{2n}} \chi_X(x) f(x+h) \overline{g(x+h)} \langle \varphi(h), \varphi(h) \rangle \, \mathrm{d}x \mathrm{d}\mu_\sigma(h). \end{aligned}$$

It follows that

$$\begin{aligned} [E(X)f](x) &= f(x) \int_{\mathbb{R}^n} \chi_X(x-h) \|\varphi(h)\|^2 \, \mathrm{d}\mu_\sigma(h) \\ &= f(x) \int_{\mathbb{R}^n} \chi_X(x-h) \mathrm{d}\mu(h), \end{aligned}$$

where $\mathrm{d}\mu(h) = \|\varphi(h)\|^2 \mathrm{d}\mu_\sigma(h)$ is a probability measure on $\mathbb{R}^n$.

The last formula can be rewritten as in eq. (4.11) in terms of the selfadjoint operator Q

$$E(X) = (\chi_X * \tilde{\mu})(Q),$$

where $\chi_X * \tilde{\mu}$ is the convolution of the function χ_X with the measure defined by

$$\int_{\mathbb{R}^n} \varphi(x) \mathrm{d}\tilde{\mu} = \int_{\mathbb{R}^n} \varphi(-x) \mathrm{d}\mu \qquad \forall \varphi \in C_c(\mathbb{R}^n).$$

4.5 Covariance under dilations

Besides covariance (4.3) and invariance (4.4), the canonical position observable Π_Q has still more symmetry properties. Namely, let $\mathbb{R}_+$ be the set of positive real numbers regarded as a multiplicative group. It has a family of unitary representations $\{A_t \mid t \in \mathbb{R}\}$ acting on $\mathcal{H}$, and given by

$$[A_t(a)f](x) = \frac{1}{\sqrt{a}} f\left(a^{-1}(x-t)+t\right).$$

It is a direct calculation to verify that for all $a \in \mathbb{R}_+, X \in \mathcal{B}(\mathbb{R})$,

$$A_0(a)\Pi_Q(X)A_0(a)^* = \Pi_Q(aX).$$

We adopt the following terminology, which we take from [22].

Definition 4.5.1 *An observable $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is* ***covariant under dilations*** *if there exists a unitary representation A of $\mathbb{R}_+$ such that for all $a \in \mathbb{R}_+$ and $X \in \mathcal{B}(\mathbb{R})$,*

$$A(a)E(X)A(a)^* = E(aX). \tag{4.14}$$

The canonical position observable Π_Q is not the only position observable which is covariant under dilations. An observable $E_{\delta_t}, t \in \mathbb{R}$, is a translated version of Π_Q, namely, for any $X \in \mathcal{B}(\mathbb{R})$,

$$E_{\delta_t}(X) = \Pi_Q(X-t) = U(t)^*\Pi_Q(X)U(t).$$

Since $A_{-t}(a) = U(t)^*A_0(a)U(t)$, the observable E_{δ_t} is covariant under dilations, with, for example, $A = A_{-t}$.

Proposition 4.5.2 *Let E be a position observable on $\mathbb{R}$. The following conditions are equivalent:*

- *(a) E is covariant under dilations;*
- *(b) $\|E(U)\| = 1$ for every nonempty open set $U \subset \mathbb{R}$;*
- *(c) $E = E_{\delta_t}$ for some $t \in \mathbb{R}$;*
- *(d) E is a sharp observable.*

Proof. Let E be covariant under dilations. In a similar way as in [22, Lemma 3] we can show that $\|E(U)\| = 1$ for all nonempty open sets U. In fact, assuming the contrary, we can find a closed interval I with nonvoid interior such that $\|E(I)\| = 1 - \varepsilon < 1$. By translational covariance of E, it is not restrictive to assume that I is centered at the origin. If $f \in \mathcal{H}$ we then have

$$\mu_{f,f}(nI) = \langle f, E(nI)f\rangle = \langle A(n)^* f, E(I)A(n)^* f\rangle \le 1-\varepsilon$$

for all $n \in \mathbb{N}$. By inner regularity of $\mu_{f,f}$, we have $\mu_{f,f}(\mathbb{R}) \le 1-\varepsilon$, hence $E(\mathbb{R}) \ne I$, which is absurd. So, (a) implies (b).

Assume then that (b) holds. For any nonempty open set U we get

$$1 = \|E(U)\| = \operatorname{ess\,sup}_{x\in\mathbb{R}}\rho(x+U). \tag{4.15}$$

It follows that $\operatorname{supp}(\rho)$ contains only one point. Indeed, assume on the contrary that $\operatorname{supp}(\rho)$ contains two points $x_1 \neq x_2$ and denote $U = \{x \in \mathbb{R} \,|\, |x| < \frac{1}{4}|x_1 - x_2|\}$. Since $x_1 + U$ and $x_2 + U$ are neighborhoods of x_1 and x_2, respectively, we have $m_i := \rho(x_i + U) > 0$ for $i = 1, 2$. Then, for any $x \in \mathbb{R}$, $\rho(x+U) \leq 1 - \min(m_1, m_2)$. This is in contradiction with (4.15). Hence, (b) implies (c).

As previously mentioned, (c) implies (a). Clearly, (c) also implies (d). Since (d) implies (b) the proof is complete. ■

The dilation covariance means that the observable in question has no scale dependence. A realistic position measurement apparatus has a limited accuracy and hence it cannot define a position observable which is covariant under dilations. Thus, sharp position observables are not suitable to describe nonideal situations.

If $E = E_{\delta_t}$, one could ask what is the most general form of the representation A of $\mathbb{R}_+$ satisfying eq. (4.14). The answer is given in the next proposition.

Proposition 4.5.3 *If A is a unitary representation of $\mathbb{R}_+$ satisfying eq. (4.14) with $E = E_{\delta_t}$, then there exists a measurable function $\beta : \mathbb{R} \longrightarrow \mathbb{T}$ such that*

$$[A(a)f](x) = \frac{1}{\sqrt{a}}\beta(x+t)\overline{\beta\left(a^{-1}(x+t)\right)}f\left(a^{-1}(x+t)-t\right).$$

In particular, A is equivalent to A_{-t}.

Proof. Let $A'(a) = U(t)A(a)U(t)^*$. Then, $A'(a)\Pi_Q(X)A'(a)^* = \Pi_Q(aX)$. Denote with Π_Q^+ the restriction of Π_Q to the Borel subsets of $\mathbb{R}_+$. Then, $S_0 = \left(A_0, \Pi_Q^+, L^2(0,+\infty)\right)$ and $S = \left(A', \Pi_Q^+, L^2(0,+\infty)\right)$ are transitive imprimitivity systems of the group $\mathbb{R}_+$ based on $\mathbb{R}_+$. By Mackey imprimitivity theorem, there exists a Hilbert space $\mathcal{K}$ such that $S = \operatorname{ind}_{\{1\}}^{\mathbb{R}_+}(I_{\mathcal{K}})$, where $I_{\mathcal{K}}$ is the trivial representation of $\{1\}$ acting in $\mathcal{K}$. Since $S_0 = \operatorname{ind}_{\{1\}}^{\mathbb{R}_+}(1)$, we have the isomorphism of intertwining operators $\mathcal{C}(1, I_{\mathcal{K}}) \simeq \mathcal{C}(S_0, S)$, and hence there exists an isometry $W_1 : L^2(0,+\infty) \longrightarrow L^2(0,+\infty)$ intertwining S_0 with S. In particular, $W_1\Pi_Q^+ = \Pi_Q^+W_1$, and hence there exists a measurable function $\beta_1 : \mathbb{R}_+ \longrightarrow \mathbb{T}$ such that

$$[W_1 f](x) = \beta_1(x)f(x) \quad \forall f \in L^2(0,+\infty).$$

It follows that W_1 is unitary.

Reasoning as above, one finds a unitary operator W_2 intertwining the restrictions of A_0 and A' to $L^2(-\infty, 0)$, with

$$[W_2 f](x) = \beta_2(x) f(x) \quad \forall f \in L^2(-\infty, 0),$$

for some measurable function $\beta_2 : \mathbb{R}_- \longrightarrow \mathbb{T}$. Then, $W = W_1 \oplus W_2$ is unitary on $L^2(-\infty, +\infty)$, and $A(a) = U(t)^* W A_0(a) W^* U(t)$ has the claimed form for all $a \in \mathbb{R}_+$. ■

4.6 State distinction power of a position observable

In this section, we investigate the ability of position observables to discriminate between different states, that is we compare the state distinction power of two position observables (see also [11]).

Definition 4.6.1 *Let E_1 and E_2 be observables on $\mathbb{R}$. The **state distinction power** of E_2 is greater than or equal to E_1 if for all $T, T' \in \mathcal{S}(\mathcal{H})$,*

$$p_T^{E_2} = p_{T'}^{E_2} \Rightarrow p_T^{E_1} = p_{T'}^{E_1}.$$

*In this case we denote $E_1 \sqsubseteq E_2$. If $E_1 \sqsubseteq E_2 \sqsubseteq E_1$ we say that E_1 and E_2 are **informationally equivalent** and denote $E_1 \sim E_2$. If $E_1 \sqsubseteq E_2$ and $E_2 \not\sqsubseteq E_1$, we write $E_1 \sqsubset E_2$.*

Example 4.6.2 *An observable $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is **trivial** if $p_T^E = p_{T'}^E$ for all states $T, T' \in \mathcal{S}(\mathcal{H})$. A trivial observable E is then of the form $E(X) = \lambda(X) I$, $X \in \mathcal{B}(\mathbb{R})$, for some probability measure λ. The state distinction power of any observable E' is greater than or equal to that of the trivial observable E. Clearly there is no trivial position observable on $\mathbb{R}$.*

Example 4.6.3 *An observable $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is called **informationally complete** if $p_T^E \neq p_{T'}^E$ whenever $T \neq T'$. The state distinction power of an informationally complete observable is greater than or equal to that of any other observable E_1 on $\mathbb{R}$. It is easy to see that there is no informationally complete position observable. Namely, let ψ be a unit vector, $p \neq 0$ a real number, and denote $\psi' = V(p)\psi$. Then the states $T = \langle \psi, \cdot \rangle \psi$ and $T' = \langle \psi', \cdot \rangle \psi'$ are different but for any position observable E_ρ, $p_T^{E_\rho} = p_{T'}^{E_\rho}$ since $V(p)$ commutes with all the effects $E_\rho(X)$, $X \in \mathcal{B}(\mathbb{R})$.*

We will next think of $\sim$ as a relation on the set $\mathcal{POS}_\mathbb{R}$. The relation $\sim$ is clearly reflexive, symmetric and transitive, and hence it is an equivalence relation. We denote the equivalence class of a position observable E by $[E]$ and the space of equivalence classes as $\mathcal{POS}_\mathbb{R}/\sim$. The relation $\sqsubseteq$ induces a partial order in the set $\mathcal{POS}_\mathbb{R}/\sim$ in a natural way.

Let E_ρ be a position observable and T a state. The probability measure $p_T^{E_\rho}$ is the convolution of the probability measures $p_T^{\Pi_Q}$ and ρ,

$$p_T^{E_\rho} = p_T^{\Pi_Q} * \rho. \tag{4.16}$$

It is clear from (4.16) that for all $T, T' \in \mathcal{S}(\mathcal{H})$,

$$p_T^{\Pi_Q} = p_{T'}^{\Pi_Q} \Rightarrow p_T^{E_\rho} = p_{T'}^{E_\rho},$$

and hence $E_\rho \sqsubseteq \Pi_Q$. We conclude that $[\Pi_Q]$ is the only maximal element of the partially ordered set $\mathcal{POS}_{\mathbb{R}}/\sim$.

It is shown in [35, Prop. 5] that a position observable E_ρ belongs to the maximal equivalence class $[\Pi_Q]$ if and only if $\operatorname{supp}(\widehat{\rho}) = \mathbb{R}$. The following proposition characterizes the equivalence classes completely.

Proposition 4.6.4 *Let ρ_1, ρ_2 be probability measures on $\mathbb{R}$ and E_{ρ_1}, E_{ρ_2} the corresponding position observables. Then*

$$E_{\rho_1} \sqsubseteq E_{\rho_2} \Longleftrightarrow \operatorname{supp}(\widehat{\rho_1}) \subseteq \operatorname{supp}(\widehat{\rho_2}).$$

Proof. Taking the Fourier transform of eq. (4.16), we get

$$\mathcal{F}(p_T^{E_\rho}) = \mathcal{F}(p_T^{\Pi_Q})\mathcal{F}(\rho). \tag{4.17}$$

Since the Fourier transform is injective, it is clear from the above relation that $\operatorname{supp}(\widehat{\rho_1}) \subseteq \operatorname{supp}(\widehat{\rho_2})$ implies $E_{\rho_1} \sqsubseteq E_{\rho_2}$.

Conversely, suppose $\operatorname{supp}(\widehat{\rho_1}) \nsubseteq \operatorname{supp}(\widehat{\rho_2})$. As $\widehat{\rho_i}$, $i = 1, 2$, are continuous functions and $\widehat{\rho_i}(\xi) = \overline{\widehat{\rho_i}(-\xi)}$, there exists a closed interval $[2a, 2b]$, with $0 \leq a < b$, such that $[2a, 2b] \cup [-2b, -2a] \subseteq \operatorname{supp}(\widehat{\rho_1})$ and $([2a, 2b] \cup [-2b, -2a]) \cap \operatorname{supp}(\widehat{\rho_2}) = \emptyset$. Define the functions

$$\begin{aligned} h_1 &= \frac{1}{\sqrt{2(b-a)}}\left(\chi_{[a,b]} - \chi_{[-b,-a]}\right), \\ h_2 &= \frac{1}{\sqrt{2(b-a)}}\left(\chi_{[a,b]} + \chi_{[-b,-a]}\right), \end{aligned}$$

and for $i = 1, 2$, denote

$$h_i^*(\xi) := \overline{h_i(-\xi)}.$$

Define

$$f_i = \mathcal{F}^{-1}(h_i),$$

and let T_i be the one-dimensional projection $|f_i\rangle\langle f_i|$. We then have

$$\mathrm{d}p_{T_i}^{\Pi_Q}(x) = |f_i(x)|^2\,\mathrm{d}x = \left|\left(\mathcal{F}^{-1}h_i\right)(x)\right|^2\mathrm{d}x = \mathcal{F}^{-1}\left(h_i * h_i^*\right)(x)\mathrm{d}x,$$

and

$$\begin{aligned}\mathcal{F}(p_{T_i}^{\Pi_Q}) &= \mathcal{F}\mathcal{F}^{-1}\left(h_i * h_i^*\right) = h_i * h_i^* \\ &= \frac{1}{2(b-a)}\Big(2\chi_{[a,b]} * \chi_{[-b,-a]} + (-1)^i \chi_{[-b,-a]} * \chi_{[-b,-a]} \\ &\quad + (-1)^i \chi_{[a,b]} * \chi_{[a,b]}\Big).\end{aligned}$$

Since

$$\begin{aligned}\operatorname{supp}\left(\chi_{[a,b]} * \chi_{[-b,-a]}\right) &= [a-b, b-a], \\ \operatorname{supp}\left(\chi_{[a,b]} * \chi_{[a,b]}\right) &= [2a, 2b], \\ \operatorname{supp}\left(\chi_{[-b,-a]} * \chi_{[-b,-a]}\right) &= [-2b, -2a],\end{aligned}$$

an application of (4.17) shows that

$$\begin{aligned}\mathcal{F}(p_{T_1}^{E_{\rho_1}}) &\neq \mathcal{F}(p_{T_2}^{E_{\rho_1}}), \\ \mathcal{F}(p_{T_1}^{E_{\rho_2}}) &= \mathcal{F}(p_{T_2}^{E_{\rho_2}}),\end{aligned}$$

or in other words, $E_{\rho_1} \not\sqsubseteq E_{\rho_2}$. ■

Remark 4.6.5 *It follows from the above proposition that* $E_1 \sqsubset E_2 \Longleftrightarrow \operatorname{supp}(\widehat{\rho_1}) \subset \operatorname{supp}(\widehat{\rho_2})$*, and hence the set* $\mathcal{POS}_{\mathbb{R}}/\sim$ *has no minimal element. Indeed, if* ρ_2 *is a probability measure, there always exists a probability measure* ρ_1 *such that* $\operatorname{supp}(\widehat{\rho_1}) \subset \operatorname{supp}(\widehat{\rho_2})$*. In fact, since* $\widehat{\rho_2}$ *is continuous,* $\widehat{\rho_2}(\xi) = \overline{\widehat{\rho_2}(-\xi)}$ *and* $\widehat{\rho_2}(0) = \rho_2(\mathbb{R}) \neq 0$*, there exists* $a > 0$ *such that the closed interval* $[-a, a]$ *is strictly contained in* $\operatorname{supp}(\widehat{\rho_2})$*. If we define* $h = \frac{1}{\sqrt{a}}\chi_{[-\frac{a}{2},\frac{a}{2}]}$*,* $f = \mathcal{F}^{-1}h$*, then* $d\rho_1(x) := |f(x)|^2 dx$ *is a probability measure, and* $\operatorname{supp}(\widehat{\rho_1}) = \operatorname{supp}(h * h) = [-a, a]$*.*

4.7 Limit of resolution of a position observable

Let $\Pi : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be a sharp observable. For any nontrivial projection $\Pi(X)$, there exist states $T, T' \in \mathcal{S}(\mathcal{H})$ such that $p_T^{\Pi}(X) = 1$ and $p_{T'}^{\Pi}(X) = 0$. Using the terminology of [11, II.2.1], we may say that $\Pi(X)$ is a *sharp property* and it is *real* in the state T.

In general, an observable E has effects as its values which are not projections and, hence, not sharp properties. An effect $E(X) \in \operatorname{ran} E$ is called **regular** if its spectrum extends both above and below $1/2$. This means that there exist states $T, T' \in \mathcal{S}(\mathcal{H})$ such that $\operatorname{tr}[TE(X)] > 1/2$ and $\operatorname{tr}[T'E(X)] < 1/2$. In this sense regular effects can be seen as *approximately realizable properties* (see again [11, II.2.1]). The observable E is called **regular** if all the nontrivial effects $E(X)$ are regular.

It is shown in [35, Prop. 4] that if a probability measure ρ is absolutely continuous with respect to the Lebesgue measure, then the position observable E_ρ is not regular. Here we modify the notion of regularity to get a quantification of sharpness, or resolution, of position observables.

For any $x \in \mathbb{R}, r \in \mathbb{R}_+$, we denote the interval $[x - r/2, x + r/2]$ by $I_{x;r}$. We also denote $I_r = I_{0;r}$. Let $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable and $\alpha > 0$. We say that E is **α-regular** if all the nontrivial effects $E(I_{x;r})$, $x \in \mathbb{R}$, $r \geq \alpha$, are regular.

Definition 4.7.1 *Let $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be an observable. We denote*

$$\gamma_E = \inf \{\alpha > 0 \mid E \textit{ is } \alpha\textit{-regular}\}$$

*and say that γ_E is the **limit of resolution of** E.*

It follows directly from definitions that the limit of resolution of a regular observable is 0. Especially, the limit of resolution of canonical position observables is 0.

Example 4.7.2 *Let E be a trivial observable (see Example 4.6.2). For any $X \in \mathcal{B}(\mathbb{R})$, we have either $E(X) \geq \frac{1}{2}I$ or $E(X) \leq \frac{1}{2}I$. Hence, $\gamma_E = \infty$.*

Proposition 4.7.3 *A position observable E_ρ is α-regular if and only if*

$$\operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(I_{x,\alpha}) > 1/2, \tag{4.18}$$

where the essential supremum is taken with respect to the Lebesgue measure of $\mathbb{R}$.

Proof. An effect $E_\rho(X)$ is regular if and only if $\|E_\rho(X)\| > 1/2$ and $\|E_\rho(\mathbb{R} \setminus X)\| > 1/2$. Since the norm of the multiplicative operator $E_\rho(X)$ is $\operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(X - x)$, we conclude that $E_\rho(X)$ is regular if and only if

$$\operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(X - x) > 1/2 \quad \text{and} \quad \operatorname{ess\,inf}_{x\in\mathbb{R}} \rho(X - x) < 1/2.$$

Thus, E_ρ is α-regular if and only if, for all $r \geq \alpha$,

$$\operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(I_{x;r}) > 1/2 \quad \text{and} \quad \operatorname{ess\,inf}_{x\in\mathbb{R}} \rho(I_{x;r}) < 1/2.$$

The second condition is always satisfied and from the first eq. (4.18) follows. ■

Corollary 4.7.4 *A position observable E_ρ has a finite limit of resolution and*

$$\gamma_{E_\rho} = \inf \{\alpha > 0 \mid \operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(I_{x;\alpha}) > 1/2\}.$$

In the next chapter, we will prove an Heisenberg-like uncertainty relation involving the limits of resolution of coexistent position and momentum observables. For the moment, we end this section with a result that characterises the position observables which are regular. It is taken from [16].

Proposition 4.7.5 *Let E_ρ be a position observable. The following conditions are equivalent:*

(i) E_ρ *is regular;*

(ii) $\gamma_{E_\rho} = 0$*;*

(iii) there exists $\overline{x} \in \mathbb{R}$ *and* $\lambda \in M_1^+(\mathbb{R})$ *with* $\overline{x} \in \mathrm{supp}\lambda$ *such that* $\rho = \frac{1}{2}\delta_{\overline{x}} + \frac{1}{2}\lambda$.

Proof. It is clear that (i) implies (ii).

Let $\gamma_{E_\rho} = 0$. This means that

$$\inf\{\alpha > 0 \mid \mathrm{ess\,sup}_{x\in\mathbb{R}}\rho(I_{x;\alpha}) > \frac{1}{2}\} = 0. \tag{4.19}$$

For every $\alpha > 0$, denote

$$A_\alpha = \{x \in \mathbb{R} \mid \rho(I_{x;\alpha}) > \frac{1}{2}\}.$$

Since $\alpha_1 > \alpha_2$ implies $A_{\alpha_1} \supset A_{\alpha_2}$, it follows from (4.19) that $A_\alpha \neq \emptyset$ for all $\alpha > 0$. For each $n = 1, 2\ldots$, choose an element $x_n \in A_{1/n}$. We have $\rho(I_{x_n,1/n}) > \frac{1}{2}$. Since ρ is a finite measure, the sequence $(x_n)_{n\geq 1}$ is bounded. Hence, there exists a subsequence $(x_{n_k})_{k\geq 1}$ converging to some $\overline{x} \in \mathbb{R}$. For each $\beta > 0$, there exists k such that $I_{x_{n_k},1/n_k} \subset I_{\overline{x},\beta}$, so that $\rho(I_{\overline{x},\beta}) > \frac{1}{2}$. Thus,

$$\rho(\{\overline{x}\}) = \rho(\cap_{\beta>0} I_{\overline{x},\beta}) = \lim_{\beta\to 0}\rho(I_{\overline{x},\beta}) \geq \frac{1}{2}.$$

It follows that $\lambda = 2\rho - \delta_{\overline{x}}$ is a probability measure. For all $\beta > 0$, we have $\lambda(I_{\overline{x},\beta}) = 2\rho(I_{\overline{x},\beta}) - 1 > 0$, which implies $\overline{x} \in \mathrm{supp}\lambda$. Thus, (ii) implies (iii).

Assume that (iii) holds. We start by noticing that E_ρ is regular if and only if $\|E_\rho(X)| = \frac{1}{2}\mathrm{ess\,sup}_{q\in\mathbb{R}}\{\delta_{x_0}(X-q) + \lambda(X-q)\} > \frac{1}{2}$ for every Borel set in which $E_\rho \neq 0, I$. Since we know that $E_\rho(X) = 0$ is equivalent to $\mu(X) = 0$ we can assume that X is not a null or co-null set. Let us begin by evaluating

$$\mathrm{ess\,sup}_{q\in\mathbb{R}}\{\delta_{x_0}(X-q) + \lambda(X-q)\} = \mathrm{ess\,sup}_{q\in\mathbb{R}}\{\chi_{X-x_0}(q) + \lambda(X-q)\}$$

in the case X is a finite measure set. It is a matter of proving that $\lambda(X-(\cdot))$ has non void essential image on the support of the characteristic function χ_{X-x_0}, that is:

$$\operatorname{ess\,sup}_{q\in X-x_0}\lambda(X-(\cdot))>0$$

This is also equivalent to prove that

$$\int_{X-x_0}\lambda(X-q)d\mu(q)>0$$

but an easy calculation shows that the last integral is equal to

$$\int_{\mathbb{R}}\chi_{X-x_0}*\check{\chi}_{X-x_0}(x)d\lambda(x)$$

where $\check{\chi}_X(x):=\chi_X(-x)$. It is a well known fact that the integrand is a continuous function (since it is the convolution of two continuous functions). Observing that $x_0\in\operatorname{supp}(\chi_{X-x_0}*\check{\chi}_{X-x_0})$, we get immediately that the integral is not zero. So we have the thesis in the case of finite measure sets.

Let us now consider the case $\mu(X)=\infty$. We must show that either $E_\rho(X)=I$ or $E_\rho(X)$ has spectrum below $\frac{1}{2}$. By a previous observation we can limit ourselves to the case $\mu(X')\neq 0$ since in the case of co-null set $E_\rho(X)=I$. Hence we can assume that X' cointains a finite measure set Y to which the above result apply. So we have

$$\left\|E_\rho(X')\right\|\geq\|E_\rho(Y)\|>\frac{1}{2}$$

Since $E_\rho(X')=I-E_\rho(X)$, denoting with $\sigma(A)$ the spectrum of the operator A, we have $\sigma(E_\rho(X))=1-\sigma(E_\rho(X'))$ and we are done. ■

4.8 Position and momentum observables on $\mathbb{R}^3$

4.8.1 Definitions

In this section, $\mathcal{H}=L^2(\mathbb{R}^3;\mathbb{C}^{2j+1})$ is the Hilbert space of a nonrelativistic particle with spin j in the Euclidean space. Let $Q_i, i=1,2,3$, denote the multiplicative operators on $\mathcal{H}$ given by $[Q_if](\vec{x})=x_if(\vec{x})$, where x_i is the ith component of $\vec{x}$. By P_i we mean the operator $\mathcal{F}^{-1}Q_i\mathcal{F}$ and we denote $\vec{Q}=(Q_1,Q_2,Q_3)$, $\vec{P}=(P_1,P_2,P_3)$. The space translation group $\mathbb{R}^3$ has a unitary representation $U(\vec{q})=e^{-i\vec{q}\cdot\vec{P}}$ and similarly, the momentum boost group has a representation $V(\vec{p})=e^{i\vec{p}\cdot\vec{Q}}$. It is an immediate observation that the sharp observables $\Pi_{\vec{Q}}$ and $\Pi_{\vec{P}}$ on $\mathbb{R}^3$, respectively associated to the representations V and U by Stone-Naimark-Ambrose-Godement theorem, satisfy the obvious covariance and invariance conditions, analogous to (4.5)-(4.8). The rotation group $SO(3)$ acts in $\mathcal{H}$ according to the projective representation D, given by

$$[D(R)f](\vec{x})=D^j\left(R'\right)f(R^{-1}\vec{x}).$$

Here, D^j is the irreducible representation of $SU(2)$ in $\mathbb{C}^{2j+1}$, and R' is an element of $SU(2)$ lying above R. It is straightforward to verify that the sharp observables $\Pi_{\vec{Q}}$ and $\Pi_{\vec{P}}$ are covariant under rotations, that is, for all $R \in SO(3)$ and $X \in \mathcal{B}(\mathbb{R}^3)$,

$$\begin{aligned} D(R)\Pi_{\vec{Q}}(X)D(R)^* &= \Pi_{\vec{Q}}(RX), \\ D(R)\Pi_{\vec{P}}(X)D(R)^* &= \Pi_{\vec{P}}(RX). \end{aligned}$$

These observations motivate to the following definitions.

Definition 4.8.1 *An observable $E : \mathcal{B}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{H})$ is a* ***position observable on*** $\mathbb{R}^3$ *if, for all $\vec{q}, \vec{p} \in \mathbb{R}^3$, $R \in SO(3)$ and $X \in \mathcal{B}(\mathbb{R}^3)$,*

$$U(\vec{q})E(X)U(\vec{q})^* = E(X + \vec{q}), \tag{4.20}$$

$$V(\vec{p})E(X)V(\vec{p})^* = E(X), \tag{4.21}$$

$$D(R)E(X)D(R)^* = E(RX). \tag{4.22}$$

We will denote by $\mathcal{POS}_{\mathbb{R}^3}$ the set of all position observables on $\mathbb{R}^3$.

Definition 4.8.2 *An observable $F : \mathcal{B}(\mathbb{R}^3) \to \mathcal{L}(\mathcal{H})$ is a* ***momentum observable on*** $\mathbb{R}^3$ *if, for all $\vec{q}, \vec{p} \in \mathbb{R}^3$, $R \in SO(3)$ and $X \in \mathcal{B}(\mathbb{R}^3)$,*

$$\begin{aligned} U(\vec{q})F(X)U(\vec{q})^* &= F(X), \\ V(\vec{p})F(X)V(\vec{p})^* &= F(X + \vec{p}), \\ D(R)F(X)D(R)^* &= F(RX). \end{aligned}$$

4.8.2 Structure of position observables on $\mathbb{R}^3$

We say that a probability measure ρ on $\mathbb{R}^3$ is rotation invariant if for all $X \in \mathcal{B}(\mathbb{R}^3)$ and $R \in SO(3)$,

$$(R \cdot \rho)(X) := \rho(R^{-1}X) \equiv \rho(X).$$

The set of rotation invariant probability measures on $\mathbb{R}^3$ is denoted by $M(\mathbb{R}^3)^{1,+}_{inv}$. Using the isomorphism $\mathbb{R}^3 \setminus \{0\} \simeq \mathbb{R}_+ \times S^2$ and the disintegration of measures, the restriction of any measure $\rho \in M(\mathbb{R}^3)^{1,+}_{inv}$ to the subset $\mathbb{R}^3 \setminus \{0\}$ can be written in the form

$$\mathrm{d}\rho|_{\mathbb{R}^3\setminus\{0\}}(\vec{r}) = \mathrm{d}\rho_{\mathrm{rad}}(r)\,\mathrm{d}\rho_{\mathrm{ang}}(\Omega),$$

where ρ_{rad} is a finite measure on $\mathbb{R}_+$ with $\rho_{\mathrm{rad}}(\mathbb{R}_+) = 1 - \rho(\{0\})$, and ρ_{ang} is the $SO(3)$-invariant measure on the sphere S^2 normalized to 1.

Given a rotation invariant probability measure ρ, the formula

$$E_\rho(X) = \int \rho(X - \vec{q})\,\mathrm{d}\Pi_{\vec{Q}}(\vec{q}), \quad X \in \mathcal{B}(\mathbb{R}^3), \tag{4.23}$$

defines a position observable on $\mathbb{R}^3$.

Proposition 4.8.3 *Any position observable E on $\mathbb{R}^3$ is of the form $E = E_\rho$ for some $\rho \in M(\mathbb{R}^3)^{1,+}_{inv}$.*

Proof. It is shown in §4.4 that if E satisfies eqs. (4.20), (4.21), then E is given by eq. (4.23) for some probability measure ρ in $\mathbb{R}^3$. If $\varphi \in C_c\left(\mathbb{R}^3\right)$, we define $E\left(\varphi\right)$ as in Remark 1.2.4. For all $f \in L^2\left(\mathbb{R}^3; \mathbb{C}^{2j+1}\right)$, define the measure

$$\mathrm{d}\mu_f(\vec{x}) = |f(\vec{x})|^2\, \mathrm{d}\vec{x}.$$

We then have

$$\langle f, E(\varphi) f\rangle = \left(\mu_f * \rho\right)(\varphi).$$

From (4.22) it then follows

$$\left(\mu_{D(R)f} * \rho\right)(\varphi) = \left(\mu_f * \rho\right)\left(\varphi^{R^{-1}}\right), \tag{4.24}$$

where $\varphi^{R^{-1}}(\vec{x}) = \varphi(R\vec{x})$. Rewriting explicitly (4.24), we then find (since D^j is unitary on $\mathbb{C}^{2j+1}$)

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(\vec{x}+\vec{y})\left|f\left(R^{-1}\vec{x}\right)\right|^2 \mathrm{d}\vec{x}\mathrm{d}\rho(\vec{y})$$
$$= \int_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^{R^{-1}}\left(\vec{x}+\vec{y}\right)\left|f\left(\vec{x}\right)\right|^2 \mathrm{d}\vec{x}\mathrm{d}\rho(\vec{y}),$$

With some computations, setting $\psi\left(\vec{x}\right) = \varphi\left(-R\vec{x}\right)$, this gives

$$\int_{\mathbb{R}^3}\left(\psi * |f|^2\right)(-\vec{y})\mathrm{d}\left(R^{-1}\cdot\rho\right)(\vec{y}) = \int_{\mathbb{R}^3}\left(\psi * |f|^2\right)(-\vec{y})\mathrm{d}\rho(\vec{y}) \tag{4.25}$$

(here $R^{-1}\cdot\rho$ denotes the measure on $\mathbb{R}^3$ such that $\left(R^{-1}\cdot\rho\right)(\phi) = \rho\left(\phi^R\right)$ for all $\phi \in C_c\left(\mathbb{R}^3\right)$). Letting ψ and $|f|$ vary in $C_c\left(\mathbb{R}^3\right)$, the functions $\psi * |f|^2$ span a dense subset of $C_0\left(\mathbb{R}^3\right)$. From eq. (4.25), it then follows that $R^{-1}\cdot\rho = \rho$. ■

Proposition 4.8.4 *Let E be a position observable on $\mathbb{R}^3$. The following facts are equivalent:*

(a) $\|E(U)\| = 1$ for every nonempty open set $U \subset \mathbb{R}$;

(b) E is a sharp observablen ;

(c) $E = \Pi_{\vec{Q}}$.

Proof. It is clear that (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). Hence, it is enough to show that (a) implies (c). As in the proof of Proposition 4.5.2, it follows from (a) that $\rho = \delta_{\vec{t}}$ for some $\vec{t} \in \mathbb{R}^3$. However, the probability measure $\delta_{\vec{t}}$ is rotation invariant if and only if $\vec{t} = \vec{0}$. This means that $E = \Pi_{\vec{Q}}$. ■

Chapter 5

Coexistence of position and momentum observables

We now conclude the discussion we have begun in the previous chapter, and we finally consider the joint observables of position and momentum, i.e. those observables defined in the phase space of the quantum system whose margins are observables of position and momentum (see definition 5.1.2 below). The covariant phase space observables described in §3.5.2 are just a subclass of the whole set of joint observables of position and momentum.

The problem of joint measurability of position and momentum observables has a long history in quantum mechanics and different viewpoints have been presented (for an overview of the subject, see e.g. [9]). In this chapter, we will show that the following remarkable fact holds: if a position observable and a momentum observable admit a joint observable, then they also admit a covariant phase space joint observable (Proposition 5.2.2). From this fact, one can derive many properties of coexistent position and momentum observables. For example, they must satisfy Heisenberg's uncertainty relation, which can be restated in different forms (see Proposition 5.2.4 and Corollary 5.2.3).

Here is a brief synopsis of this chapter. In section 5.1 we recall some concepts which are essential for our investigation. In section 5.2 we characterize those pairs of position and momentum observables which are functionally coexistent and can thus be measured jointly. Also some consequences on the properties of joint observables are investigated. In Section 5.3 we present an observation on the general problem of coexistence of position and momentum observables.

The material in this chapter is contained in [15].

5.1 Coexistence and joint observables

The notions of coexistence, functional coexistence and joint observables are essential when joint measurability of quantum observables is analyzed. We next shortly recall the definitions of these concepts. For further details we refer to a convenient survey [45] and to the references given there.

Definition 5.1.1 *Let $(\Omega_i, \mathcal{A}(\Omega_i))$, $i = 1, 2$, be measurable spaces and let $E_i : \mathcal{A}(\Omega_i) \longrightarrow \mathcal{L}(\mathcal{H})$ be observables.*

*(i) E_1 and E_2 are **coexistent** if there is a measurable space $(\Omega, \mathcal{A}(\Omega))$ and an observable $G : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ such that*

$$\mathrm{ran}\,(E_1) \cup \mathrm{ran}\,(E_2) \subseteq \mathrm{ran}\,(G).$$

*(ii) E_1 and E_2 are **functionally coexistent** if there is a measurable space $(\Omega, \mathcal{A}(\Omega))$, an observable $G : \mathcal{A}(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$ and measurable functions $f_1 : \Omega \longrightarrow \Omega_1$, $f_2 : \Omega \longrightarrow \Omega_2$, such that, for any $X_1 \in \mathcal{A}(\Omega_1)$, $X_2 \in \mathcal{A}(\Omega_2)$,*

$$E_1(X_1) = G(f_1^{-1}(X_1)), \qquad E_2(X_2) = G(f_2^{-1}(X_2)).$$

Functionally coexistent observables are clearly coexistent, but it is an open question if the reverse holds.

We now confine our discussion to observables on $\mathbb{R}$.

Definition 5.1.2 *Let $E_1, E_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be observables. An observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is their **joint observable** if for all $X, Y \in \mathcal{B}(\mathbb{R})$,*

$$\begin{aligned} E_1(X) &= G(X \times \mathbb{R}), \\ E_2(Y) &= G(\mathbb{R} \times Y). \end{aligned}$$

*In this case E_1 and E_2 are the **margins** of G.*

For observables E_1 and E_2 defined on $\mathcal{B}(\mathbb{R})$ the existence of a joint observable is equivalent to their functional coexistence. These conditions are also equivalent to the *joint measurability* of E_1 and E_2 in the sense of the quantum measurement theory (see Section 7 in [45]).

The **commutation domain** of observables E_1 and E_2, denoted by $\mathrm{com}\,(E_1, E_2)$, is the closed subspace of $\mathcal{H}$ defined as

$$\mathrm{com}\,(E_1, E_2) = \{\psi \in \mathcal{H} \mid E_1(X)E_2(Y)\psi - E_2(Y)E_1(X)\psi = 0 \ \forall X, Y \in \mathcal{B}(\mathbb{R})\}.$$

If E_1 and E_2 are sharp observables, then E_1 and E_2 are coexistent if and only if they are functionally coexistent and this is the case exactly when $\mathrm{com}\,(E_1, E_2) = \mathcal{H}$. In general, for two observables E_1 and E_2 the condition com $(E_1, E_2) = \mathcal{H}$ is sufficient but not necessary for the functional coexistence of E_1 and E_2.

In conclusion, given a pair of observables on $\mathbb{R}$ one may pose the questions of their commutativity, functional coexistence, and coexistence, in the order of increasing generality.

For position and momentum observables, the following known fact holds [10], which will be needed later. For completeness we give a proof of it.

Proposition 5.1.3 *A position observable E_ρ and a momentum observable F_ν are totally noncommutative, that is,* com $(E_\rho, F_\nu) = \{0\}$.

Proof. It is shown in [12] and [57] that for functions $f, g \in L^\infty(\mathbb{R})$ the equation

$$f(Q)g(P) - g(P)f(Q) = O$$

holds if and only if one of the following is satisfied: (i) either $f(Q)$ or $g(P)$ is a multiple of the identity operator, (ii) f and g are both periodic with minimal periods a, b satisfying $2\pi/ab \in \mathbb{Z} \setminus \{0\}$.

Let $X \subset \mathbb{R}$ be a bounded interval. Then the operators $E_\rho(X)$ and $F_\nu(X)$ are not multiples of the identity operator. Indeed, let us assume, in contrary, that $E_\rho(X) = cI$ for some $c \in \mathbb{R}$. Denote $a = 2\mu(X)$. Then the sets $X + na$, $n \in \mathbb{Z}$, are pairwisely disjoint and

$$\begin{aligned} I &\geq E_\rho(\cup_{n\in\mathbb{Z}}(X + na)) = \sum_{n\in\mathbb{Z}} E_\rho(X + na) \\ &= \sum_{n\in\mathbb{Z}} U(na)E_\rho(X)U(na)^* = \sum_{n\in\mathbb{Z}} cI. \end{aligned}$$

This means that $c = 0$. However, since $\mu(X) > 0$, we have

$$O \neq E_\rho(X) = cI = O.$$

Thus, $E_\rho(X)$ is not a multiple of the identity operator. Moreover, since $\rho(\mathbb{R}) = 1$, the function $q \mapsto \rho(X - q)$ is not periodic. We conclude that, by the above mentioned result, the operators $E_\rho(X)$ and $F_\nu(X)$ do not commute and hence, com $(E_\rho, F_\nu) \neq \mathcal{H}$.

Assume then that there exists $\psi \neq 0$, $\psi \in$ com (E_ρ, F_ν). Using the symmetry properties (4.5), (4.6), (4.7) and (4.8), a short calculation shows that for any $q, p \in \mathbb{R}$, $U(q)V(p)\psi \in$ com (E_ρ, F_ν). This implies that com (E_ρ, F_ν) is invariant under the irreducible projective representation W defined in (3.17) (recall that $W(q,p) = e^{iqp/2}U(q)V(p)$). It follows that we have either com $(E_\rho, F_\nu) = \{0\}$ or com $(E_\rho, F_\nu) = \mathcal{H}$. Since the latter possibility is ruled out, this completes the proof. ■

5.2 Joint observables of position and momentum

Looking at the symmetry conditions (4.5), (4.6), (4.7), (4.8), and recalling the definition (3.17) of the projective unitary representation W of the group of phase space translations, it is clear that an observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ has a position observable and a momentum observable as its margins if and only if, for all $q, p \in \mathbb{R}$ and $X, Y \in \mathcal{B}(\mathbb{R})$, the following conditions hold:

$$W(q,p)G(X \times \mathbb{R})W(q,p)^* = G(X \times \mathbb{R} + (q,p)), \tag{5.1}$$
$$W(q,p)G(\mathbb{R} \times Y)W(q,p)^* = G(\mathbb{R} \times Y + (q,p)). \tag{5.2}$$

(recall that $W(q,p) = e^{iqp/2}U(q)V(p)$).

On the other hand, a covariant phase space observable is a POM $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ such that, for all $q, p \in \mathbb{R}$ and $Z \in \mathcal{B}(\mathbb{R}^2)$,

$$W(q,p)G(Z)W(q,p)^* = G(Z + (q,p)), \tag{5.3}$$

and it is trivial that (5.3) implies (5.1) and (5.2). Hence, a covariant phase space observable is a joint observable of some position and momentum observables. To our knowledge, it is an open question whether (5.1) and (5.2) imply (5.3).

As proved in §3.5.2, each covariant phase space observable is of the form $G = G_T$, with

$$G_T(Z) = \frac{1}{2\pi} \int_Z W(q,p)TW(q,p)^* \, \mathrm{d}q\mathrm{d}p \quad Z \in \mathcal{B}(\mathbb{R}^2),$$

for some $T \in \mathcal{S}(\mathcal{H})$. Moreover, the correspondence $T \longmapsto G_T$ is injective from $\mathcal{S}(\mathcal{H})$ onto the set of the covariant phase space observables (see Theorem 3.4.2).

Let G_T be a covariant phase space observable and let $\sum_n \lambda_n \langle \cdot, \varphi_n \rangle \varphi_n$ be the spectral decomposition of the state T. With an easy computation, one finds that the margins of G_T are the position observable E_ρ and a the momentum observable F_ν with

$$\mathrm{d}\rho(q) = e(q)\mathrm{d}q, \quad e(q) = \sum\nolimits_n \lambda_n |\varphi_n(-q)|^2, \tag{5.4}$$
$$\mathrm{d}\nu(p) = f(p)\mathrm{d}p, \quad f(p) = \sum\nolimits_n \lambda_n |\widehat{\varphi_n}(-p)|^2. \tag{5.5}$$

The form of ρ and ν in (5.4) and (5.5) implies that, in general, the margins E_ρ and F_ν do not determine G_T, that is, another covariant phase space observable $G_{T'}$ may have the same margins. Indeed, the functions $|\varphi(\cdot)|$ and $|\hat{\varphi}(\cdot)|$ do not define the vector φ uniquely up to a phase factor. Here we provide an example in which this phenomenon occurs.

Example 5.2.1 *Consider the functions*

$$\varphi_{a,b}(q) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-(a+ib)q^2},$$

with $a, b \in \mathbb{R}$ and $a > 0$. The Fourier transform of $\varphi_{a,b}$ is

$$\begin{aligned} \hat{\varphi}_{a,b}(p) &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} \exp\left(-\frac{ap^2}{4(a^2+b^2)}\right) \\ &\quad \times \exp\left(\frac{ibp^2}{4(a^2+b^2)} - \frac{i}{2}\arctan\frac{b}{a}\right). \end{aligned}$$

For $b \neq 0$, we see that $T_1 = \langle \varphi_{a,b}, \cdot \rangle \varphi_{a,b}$ and $T_2 = \langle \varphi_{a,-b}, \cdot \rangle \varphi_{a,-b}$ are different, but the margins of G_{T_1} and G_{T_2} are the same position and momentum observables E_ρ and F_ν, with

$$\begin{aligned} d\rho(q) &= \left(\frac{2a}{\pi}\right)^{1/2} e^{-2aq^2}\, dq \\ d\nu(p) &= \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/2} \exp\left(-\frac{ap^2}{2(a^2+b^2)}\right) dp \end{aligned}$$

As ρ and ν in (5.4) and (5.5) arise from the same state T, a multitude of uncertainty relations can be derived for the observables E_ρ and F_ν. One of the most common uncertainty relation is in terms of variances. Namely, let $\mathrm{Var}\,(p)$ denote the variance of a probability measure p,

$$\mathrm{Var}\,(p) = \int \left(x - \int x \mathrm{d}p(x)\right)^2 \mathrm{d}p(x).$$

Then for any state S,

$$\mathrm{Var}\,(p_S^{E_\rho})\mathrm{Var}\,(p_S^{F_\nu}) \geq 1 \tag{5.6}$$

(see e.g. [11], Section III.2.4 or [50], Section 5.4.) The lower bound in (5.6) can be achieved only if

$$\mathrm{Var}\,(\rho)\mathrm{Var}\,(\nu) = \frac{1}{4}, \tag{5.7}$$

and it is well known ([32], [51]) that (5.7) holds if and only if $T = \langle \cdot, \varphi \rangle \varphi$ and φ is a function of the form

$$\varphi(q) = (2a/\pi)^{1/4} e^{ibq} e^{-a(q-c)^2}, \quad a > 0,\ b, c \in \mathbb{R}.$$

It is also easily verified that, if T is as above, choosing $S = T$ the equality in (5.6) is indeed obtained.

The following proposition is the main result in this chapter, since it gives a complete characterisation of jointly measurable position and momentum observables.

Proposition 5.2.2 *Let E_ρ be a position observable and F_ν a momentum observable. If E_ρ and F_ν have a joint observable, then they also have a joint observable which is a covariant phase space observable.*

The proof of Proposition 5.2.2 is given in section 5.3, since it needs some notations and the introduction of some additional mathematical concepts. It is a rearrangement of a similar proof given by Werner in [55].

We now describe some consequences of Proposition 5.2.2. Corollaries 5.2.3 and 5.2.4 are two different restatements of Heisenberg uncertainty relation. Corollary 5.2.4 is Proposition 6 of [14] slightly rearranged.

Corollary 5.2.3 *A position observable E_ρ and a momentum observable F_ν are functionally coexistent if and only if there is a state $T \in \mathcal{S}(\mathcal{H})$ such that ρ and ν are given by (5.4) and (5.5). Especially, the uncertainty relation (5.6) is a necessary condition for the functional coexistence, and thus for the joint measurability of E_ρ and E_ν.*

Corollary 5.2.4 *Let E_ρ be a position observable and F_ν a momentum observable. If E_ρ and F_ν are functionally coexistent, then the product of the respective limits of resolutions satisfies the inequality*

$$\gamma_{E_\rho} \cdot \gamma_{F_\nu} \geq 3 - 2\sqrt{2}. \tag{5.8}$$

Proof. Since E_ρ and F_ν have a covariant phase space observable as a joint observable, there is a vector valued function $\theta \in L^2(\mathbb{R}, \mathcal{H})$ such that $\mathrm{d}\rho(q) = \|\theta(q)\|^2_{\mathcal{H}} \mathrm{d}q$ and $\mathrm{d}\nu(p) = \|\hat{\theta}(p)\|^2_{\mathcal{H}} \mathrm{d}p$. In fact, let $(f_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$. Then, with the notations of eqs. (5.4) and (5.5), the function θ is

$$\theta(q) = \sum\nolimits_n \lambda_n^{1/2} \varphi_n(-q) f_n.$$

By Proposition 4.7.3 the observable E_ρ is α-regular if and only if

$$\operatorname{ess\,sup}_{x\in\mathbb{R}} \rho\left(I_{x;\alpha}\right) > 1/2.$$

Since the map $x \longmapsto \rho\left(I_{x;\alpha}\right)$ is continuous, this is equivalent to

$$\sup\nolimits_{x\in\mathbb{R}} \rho\left(I_{x;\alpha}\right) = \sup\nolimits_{x\in\mathbb{R}} \int_{I_{x;\alpha}} \|\theta(x)\|^2_{\mathcal{H}} \, \mathrm{d}x > 1/2.$$

By the same argument, F_ν is β-regular if and only if

$$\sup\nolimits_{\xi\in\mathbb{R}} \nu\left(I_{\xi;\beta}\right) = \sup\nolimits_{\xi\in\mathbb{R}} \int_{I_{\xi;\beta}} \|\hat{\theta}(\xi)\|^2_{\mathcal{H}} \mathrm{d}\xi > 1/2.$$

Using [29, Theorem 2] extended to the case of vector valued functions, we find

$$\alpha \cdot \beta \geq 3 - 2\sqrt{2},$$

and hence (5.8) follows. ■

Corollary 5.2.5 *If a position observable E_ρ and a momentum observable F_ν are functionally coexistent, then neither E_ρ nor F_ν are regular.*

Proof. This is an immediate consequence of Corollary 5.2.3 and of item (iii) in Proposition 4.7.5. ■

We end this section with an observation about a (lacking) localization property of a joint observable of position and momentum observables. We wish to emphasize again that G in Proposition 5.2.6 is not assumed to be a covariant phase space observable.

Proposition 5.2.6 *Let G be a joint observable of a position observable E_ρ and a momentum observable F_ν and let $Z \in \mathcal{B}(\mathbb{R}^2)$ be a bounded set. Then*

(i) $\|G(Z)\| \neq 1$;

(ii) there exists a number $k_Z < 1$ such that for any $T \in \mathcal{S}(\mathcal{H})$, $p_T^G(Z) \leq k_Z$.

Proof. (i) It follows from Proposition 5.2.2 and the Paley-Wiener Theorem that either ρ or ν has unbounded support. Let us assume that, for instance, ρ has unbounded support.

Let $Z \in \mathcal{B}(\mathbb{R}^2)$ be a bounded set. Then the closure $\bar{Z}$ is compact and also the set

$$X := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : (x, y) \in \bar{Z}\} \subset \mathbb{R}$$

is compact. Since

$$\|G(Z)\| \leq \|G(X \times \mathbb{R})\| = \|E_\rho(X)\|$$

and

$$\|E_\rho(X)\| = \operatorname{ess\,sup}_{x\in\mathbb{R}} \rho(X - x) \leq \sup_{x\in\mathbb{R}} \rho(X - x),$$

it is enough to show that

$$\sup_{x\in\mathbb{R}} \rho(X - x) < 1. \tag{5.9}$$

Let us suppose, in contrary, that

$$\sup_{x\in\mathbb{R}} \rho(X - x) = 1 \tag{5.10}$$

This means that there exists a sequence $(x_n)_{n\geq 1} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \rho(X - x_n) = 1. \tag{5.11}$$

Since $\rho(\mathbb{R}) = 1$ and X is a bounded set, the sequence $(x_n)_{n\geq 1}$ is also bounded. It follows that $B := \bigcup_{n=1}^{\infty} X - x_n$ is a bounded set and by (5.11) we have $\rho(B) = 1$. This is in contradiction with the assumption that ρ has an unbounded support. Hence, (5.10) is false and (5.9) follows.

(ii) From (i) it follows that

$$1 > k_Z := \|G(Z)\| = \sup\{\langle \psi, G(Z)\psi \rangle \mid \psi \in \mathcal{H}, \|\psi\| = 1\}.$$

Let $T \in \mathcal{S}(\mathcal{H})$ and let $\sum_i \lambda_i \langle \psi_i, \cdot \rangle \psi_i$ be the spectral decomposition of T. Then

$$p_T^G(Z) = \text{tr}\,[TG(Z)] = \sum_i \lambda_i \langle \psi_i, G(Z)\psi_i \rangle \leq k_Z.$$

■

5.3 Coexistence of position and momentum observables

Since coexistence is, a priori, a more general concept than functional coexistence, we are still left with the problem of characterizing coexistent pairs of position and momentum observables. In lack of a general result we close our investigation with some observations on this problem.

Proposition 5.3.1 *Let E_ρ be a position observable and F_ν a momentum observable. If* $\text{com}\,(E_\rho) \cup \text{com}\,(F_\nu)$ *contains a nontrivial projection (not equal to O or I), then E_ρ and F_ν are not coexistent.*

Proof. Let us assume, in contrary, that there exists an observable G such that $\text{com}\,(E_\rho) \cup \text{com}\,(F_\nu) \subseteq \text{com}\,(G)$. Suppose, for instance, that $E_\rho(X)$ is a nontrivial projection. Then $E_\rho(X)$ commutes with all operators in the range of G (see e.g. [46]). In particular, $E_\rho(X)$ commutes with all $F_\nu(X)$, $X \in \mathcal{B}(\mathbb{R})$. However, this is impossible by the result proved in [12] and [57] (see also the beginning of the proof of Proposition 5.1.3). ■

Corollary 5.3.2 *If E and F are position and momentum observables, and either one of them is sharp, then E and F are not coexistent.*

Proof of Proposition 5.2.2

In order to prove Proposition 5.2.2 we need some general results about means on topological spaces, and for readers convenience they are briefly reviewed. The following material is based on [39], Chapter IV, §17, and [55].

Let Ω be a locally compact separable metric space with a metric d. By $BC\,(\Omega)$ we denote the Banach space of complex valued bounded continuous functions on Ω, with the uniform norm $\|f\|_\infty = \sup_{x\in\Omega} |f\,(x)|$. The linear subspace of continuous functions with compact support is denoted by $C_c\,(\Omega)$. Adding the index ^{r} we denote the subsets of real functions in $BC\,(\Omega)$ or in $C_c\,(\Omega)$. With the index $^+$ we denote the subsets of positive functions.

Definition 5.3.3 *A* mean *on Ω is a linear functional*

$$m : BC(\Omega) \longrightarrow \mathbb{C}$$

such that:

(i) $m(f) \geq 0$ *if* $f \in BC^+(\Omega)$*;*

(ii) $m(1) = 1$.

For a mean m on Ω we denote

$$m(\infty) = 1 - \sup\left\{m(f) \mid f \in C_c^+(\Omega),\ f \leq 1\right\}.$$

Let m be a mean on Ω. By the Riesz Representation Theorem, there exists a unique positive Borel measure m_0 on Ω such that

$$m(f) = \int_\Omega f(x)\,\mathrm{d}m_0(x) \quad \forall f \in C_c(\Omega).$$

By inner regularity of m_0 we have

$$m_0(\Omega) = \sup\left\{m(f) \mid f \in C_c^+(\Omega),\ f \leq 1\right\} = 1 - m(\infty) \leq 1.$$

Especially, any function in $BC(\Omega)$ is integrable with respect to m_0. For any $f \in BC(\Omega)$, we use the abbreviated notation

$$m_0(f) := \int_\Omega f(x)\,\mathrm{d}m_0(x).$$

Proposition 5.3.4 *If* $m(\infty) = 0$, *then*

$$m(f) = m_0(f) \quad \forall f \in BC(\Omega).$$

Proof. We fix a point $x_0 \in \Omega$. For all $R > 0$ we define

$$g_R(x) = \begin{cases} 1 & \text{if} \quad d(x_0, x) \leq R/2, \\ 3/2 - d(x_0, x)/R & \text{if} \quad R/2 < d(x_0, x) \leq 3R/2, \\ 0 & \text{if} \quad d(x_0, x) > 3R/2. \end{cases}$$

Then $g_R \in C_c^+(\Omega)$ and $g_R \leq 1$. Moreover, for any $f \in C_c^+(\Omega)$ such that $f \leq 1$ there exists $R > 0$ such that $f \leq g_R$, and hence

$$1 = \sup\left\{m(f) \mid f \in C_c^+(\Omega),\ f \leq 1\right\} = \lim_{R\to\infty} m(g_R).$$

Let $f \in BC^+(\Omega)$ and $R > 0$. Since $g_R f \in C_c(\Omega)$, we have

$$m(f) = m_0(g_R f) + m((1 - g_R) f). \tag{5.12}$$

We have $0 \leq g_R f \leq f$, f is m_0-integrable and $\lim_{R\to\infty} g_R(x) f(x) = f(x)$ for all $x \in \Omega$. Therefore, by the dominated convergence theorem we have

$$\lim_{R\to\infty} \int_\Omega g_R(x) f(x) \, \mathrm{d}m_0(x) = \int_\Omega f(x) \, \mathrm{d}m_0(x) .$$

For the other term in the sum (5.12), we have

$$m((1-g_R) f) \leq \|f\|_\infty m(1-g_R) \underset{R\to\infty}{\longrightarrow} \|f\|_\infty m(\infty) = 0.$$

Taking the limit $R \to \infty$ in (5.12) we then get

$$m(f) = m_0(f) .$$

If $f \in BC(\Omega)$, we write $f = f_1 + i f_2$ with $f_1, f_2 \in BC^r(\Omega)$, and $f_i = f_i^+ - f_i^-$ with $f_i^\pm = \frac{1}{2}(|f_i| \pm f_i) \in BC^+(\Omega)$, and we use the previous result to obtain the conclusion. ■

Let $i \in \{1, 2\}$. For $f \in BC(\Omega)$ we define

$$\widetilde{f}_i(x_1, x_2) := f(x_i) \quad \forall x_1, x_2 \in \Omega.$$

Clearly, $\widetilde{f}_i \in BC(\Omega \times \Omega)$. For a mean $m : BC(\Omega \times \Omega) \longrightarrow \mathbb{C}$, we then define

$$m_i(f) := m\left(\widetilde{f}_i\right) \quad \forall f \in BC(\Omega) .$$

The linear functional $m_i : BC(\Omega) \longrightarrow \mathbb{C}$ is a mean on Ω, which we call the ith *margin* of m.

Proposition 5.3.5 *Let m be a mean on $\Omega \times \Omega$. If $m_1(\infty) = m_2(\infty) = 0$, then $m(\infty) = 0$.*

Proof. For all $R > 0$, we define the function $g_R \in C_c(\Omega)$ as in the proof of Proposition 5.3.4. We set

$$h_R(x_1, x_2) = g_R(x_1) g_R(x_2) .$$

Clearly, $h_R \in C_c^+(\Omega \times \Omega)$, and, if $h \in C_c^+(\Omega \times \Omega)$ and $h \leq 1$, there exists $R > 0$ such that $h \leq h_R$. Since

$$\begin{aligned} 1 - h_R(x_1, x_2) &= (1 - g_R(x_1)) + g_R(x_1)(1 - g_R(x_2)) \\ &\leq (1 - g_R(x_1)) + (1 - g_R(x_2)) , \end{aligned}$$

we have

$$m(1 - h_R) \leq m_1(1 - g_R) + m_2(1 - g_R) ,$$

and the thesis follows from

$$\begin{aligned} m(\infty) &= 1 - \lim_{R\to\infty} m(h_R) \leq \lim_{R\to\infty} m_1(1 - g_R) + \lim_{R\to\infty} m_2(1 - g_R) \\ &= m_1(\infty) + m_2(\infty) = 0. \end{aligned}$$

■

For a positive Borel measure m_0 on $\Omega \times \Omega$, we denote by $(m_0)_i$, $i = 1, 2$, the two measures on Ω which are margins of m_0.

Proposition 5.3.6 *Let m be a mean on $\Omega \times \Omega$. If $m(\infty) = 0$, then $(m_0)_i = (m_i)_0$.*

Proof. Let $f \in BC(\Omega)$. By Proposition 5.3.4 we have

$$m_0\left(\widetilde{f_i}\right) = m\left(\widetilde{f_i}\right).$$

Using this equality and the definitions of $(m_0)_i$ and $(m_i)_0$ we get

$$(m_0)_i(f) = m_0\left(\widetilde{f_i}\right) = m\left(\widetilde{f_i}\right) = m_i(f) = (m_i)_0(f).$$

■

Definition 5.3.7 *An* operator valued mean *on Ω is a linear functional*

$$M : BC(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$$

such that:

(i) $M(f) \geq O$ *if* $f \in BC^+(\Omega)$*;*

(ii) $M(1) = I$.

For an operator valued mean M on Ω we denote

$$M(\infty) = I - \mathrm{LUB}\left\{M(f) \mid f \in C_c^+(\Omega),\ f \leq 1\right\}.$$

The least upper bound in Definition 5.3.7 exists by virtue of Proposition 1 in [7].

Let M be an operator valued mean on Ω. For each $f \in BC^r(\Omega)$, we have

$$M(f - \|f\|_\infty 1) \leq O, \quad M(f + \|f\|_\infty 1) \geq O.$$

It follows that

$$\|M(f)\| \leq \|f\|_\infty.$$

By Theorem 19 in [7], there exists a unique positive operator measure M_0 on Ω such that

$$M(f) = \int_\Omega f(x)\,\mathrm{d}M_0(x) \quad \forall f \in C_c(\Omega),$$

where the integral is understood in the weak sense. Similarly to the scalar case we have

$$M_0(\Omega) = I - M(\infty) \leq I, \tag{5.13}$$

and, for any $f \in BC(\Omega)$ we define

$$M_0(f) := \int_\Omega f(x)\, \mathrm{d}M_0(x).$$

Given an operator valued mean M on Ω and a unit vector $\psi \in \mathcal{H}$, we set

$$m_\psi(f) := \langle \psi, M(f)\psi \rangle \quad \forall f \in BC(\Omega).$$

It is clear that m_ψ is a mean on Ω. By Proposition 1 in [7],

$$m_\psi(\infty) = \langle \psi, M(\infty)\psi \rangle.$$

Proposition 5.3.8 *If $M(\infty) = O$, then*

$$M(f) = M_0(f) \quad \forall f \in BC(\Omega).$$

Proof. For a unit vector $\psi \in \mathcal{H}$ and a function $f \in C_c(\Omega)$, we have by definitions

$$(m_\psi)_0(f) = \langle \psi, M_0(f)\psi \rangle,$$

and this equality is valid also for any $f \in BC(\Omega)$. Since

$$m_\psi(\infty) = \langle \psi, M(\infty)\psi \rangle = 0,$$

it follows from Proposition 5.3.4 that the functional m_ψ on $BC(\Omega)$ coincides with integration with respect to the measure $(m_\psi)_0$. If $f \in BC(\Omega)$, we then have

$$\langle \psi, M_0(f)\psi \rangle = (m_\psi)_0(f) = m_\psi(f) = \langle \psi, M(f)\psi \rangle,$$

and the thesis follows. ■

The margins M_1 and M_2 of an operator valued mean M on $\Omega \times \Omega$ are defined in an analogous way as in the case of scalar means.

Proposition 5.3.9 *Let M be an operator valued mean on $\Omega \times \Omega$.*

(i) If $M_1(\infty) = M_2(\infty) = O$, then $M(\infty) = O$;

(ii) If $M(\infty) = O$, then $(M_0)_i = (M_i)_0$.

Proof. (i) Let $\psi \in \mathcal{H}$ be a unit vector. We have, by definitions, $(m_\psi)_i(f) = \langle \psi, M_i(f)\psi \rangle$ $\forall f \in BC(\Omega)$ and $(m_\psi)_i(\infty) = \langle \psi, M_i(\infty)\psi \rangle$. It follows from Proposition 5.3.5 that $m_\psi(\infty) = 0$. Since this is true for any unit vector, $M(\infty) = O$. The proof of (ii) is similar. ■

With these results we are ready to prove Proposition 5.2.2.

Proof of Proposition 5.2.2. Given a function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ and $(q,p) \in \mathbb{R} \times \mathbb{R}$, we denote by $f^{(q,p)}$ the translate of f,

$$f^{(q,p)}(x,y) := f(x+q, y+p) \quad \forall x, y \in \mathbb{R}.$$

Since $\mathbb{R} \times \mathbb{R}$ (with addition) is an Abelian group, there exists a mean m on $\mathbb{R} \times \mathbb{R}$ such that

$$m\left(f^{(q,p)}\right) = m(f)$$

for all $f \in BC(\mathbb{R} \times \mathbb{R})$ and $(q,p) \in \mathbb{R} \times \mathbb{R}$, (see [39], Theorem IV.17.5).

Let M_0 be a joint observable of E_ρ and F_ν. For each $f \in BC(\mathbb{R} \times \mathbb{R})$, for all $\varphi, \psi \in \mathcal{H}$ and $q, p \in \mathbb{R}$ we define

$$\Theta[f;\varphi,\psi](q,p) := \left\langle W(q,p)^* \varphi, M_0\left(f^{(q,p)}\right) W(q,p)^* \psi \right\rangle.$$

Since

$$\left\| M_0\left(f^{(q,p)}\right)\right\| \leq \left\| f^{(q,p)} \right\|_\infty = \|f\|_\infty$$

and $W(q,p)$ is a unitary operator, we have

$$|\Theta[f;\varphi,\psi](q,p)| \leq \|f\|_\infty \|\varphi\| \|\psi\|$$

and hence, $\Theta[f;\varphi,\psi]$ is a bounded function. We claim that $\Theta[f;\varphi,\psi]$ is continuous. Since

$$\Theta[f;\varphi,\psi](x+q,y+p) = \Theta\left[f^{(q,p)}; W(q,p)^*\varphi, W(q,p)^*\psi\right](x,y),$$

it is sufficient to check continuity at $(0,0)$. We have

$$\begin{aligned}
&|\Theta[f;\varphi,\psi](q,p) - \Theta[f;\varphi,\psi](0,0)| \\
&\quad \leq \left|\left\langle W(q,p)^*\varphi, M_0\left(f^{(q,p)}\right)\left(W(q,p)^*\psi - \psi\right)\right\rangle\right| \\
&\qquad + \left|\left\langle \left(W(q,p)^*\varphi - \varphi\right), M_0\left(f^{(q,p)}\right)\psi\right\rangle\right| \\
&\qquad + \left|\left\langle \varphi, M_0\left(f^{(q,p)} - f\right)\psi\right\rangle\right| \\
&\quad \leq \|f\|_\infty \left(\|\varphi\| \|W(q,p)^*\psi - \psi\| + \|W(q,p)^*\varphi - \varphi\| \|\psi\|\right) \\
&\qquad + \left|\left\langle \varphi, M_0\left(f^{(q,p)} - f\right)\psi\right\rangle\right|.
\end{aligned}$$

As $(q,p) \to (0,0)$, the first two terms go to 0 by strong continuity of W, and the third by the dominated convergence theorem. We have thus shown that $\Theta[f;\varphi,\psi] \in BC(\mathbb{R} \times \mathbb{R})$.

For each $f \in BC(\mathbb{R} \times \mathbb{R})$ we can then define a linear bounded operator $M^{av}(f)$ by

$$\langle \varphi, M^{av}(f)\psi\rangle := m(\Theta[f;\varphi,\psi]).$$

It is also immediately verified that the correspondence $M^{av} : BC(\mathbb{R}\times\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ is an operator valued mean on $\mathbb{R}\times\mathbb{R}$, and a short calculation shows that

$$M^{av}\left(f^{(q,p)}\right) = W(q,p)^{*}\, M^{av}(f)\, W(q,p). \tag{5.14}$$

If $f \in BC(\mathbb{R})$ and $(q,p) \in \mathbb{R}\times\mathbb{R}$, we have

$$\begin{aligned} \Theta\left[\widetilde{f}_1;\varphi,\psi\right](q,p) &= \left\langle W(q,p)^{*}\varphi, M_0\left(\widetilde{f}_1^{(q,p)}\right) W(q,p)^{*}\psi\right\rangle \\ &= \left\langle W(q,p)^{*}\varphi, W(q,p)^{*} E_\rho(f)\, W(q,p)\, W(q,p)^{*}\psi\right\rangle \\ &= \left\langle \varphi, E_\rho(f)\,\psi\right\rangle. \end{aligned}$$

(Especially, $\Theta\left[\widetilde{f}_1;\varphi,\psi\right]$ is a constant function.) Similarly,

$$\Theta\left[\widetilde{f}_2;\varphi,\psi\right](q,p) = \left\langle \varphi, F_\nu(f)\,\psi\right\rangle.$$

It follows that

$$\begin{aligned} M_1^{av}(f) &= E_\rho(f), \\ M_2^{av}(f) &= F_\nu(f). \end{aligned}$$

Since $E_\rho(\mathbb{R}) = F_\nu(\mathbb{R}) = I$, (5.13) shows that

$$M_1^{av}(\infty) = M_2^{av}(\infty) = O.$$

This together with Proposition 5.3.9 implies that $M_0^{av}(\mathbb{R}\times\mathbb{R}) = I$ and

$$\begin{aligned} (M_0^{av})_1 &= E_\rho, \\ (M_0^{av})_2 &= F_\nu. \end{aligned}$$

By (5.14) the observable M_0^{av} satisfies covariance condition (5.3). ∎

Chapter 6

Conclusions

As we have seen in chapters 2 and 3, if G is a topological group and H is a closed subgroup of G, by generalised imprimitivity theorem the classification of the POM's based on G/H and covariant with respect to a fixed representation of G is strictly related to the problem of diagonalising the representations of G induced from H. This is an highly nontrivial problem, and a satisfactory solution can be achieved only assuming that G and H are of particular type, as we have in fact done in chapters 2 and 3.

For example, if G is abelian, things are enormously simplified by the fact that the dual $\widehat{G}$ of G is itself an abelian group having the same topological properties of G. In addition, $\widehat{G/H}$ and $\widehat{H}$ are identified as subgroups or quotients of $\widehat{G}$. These are the essential features which allow to construct the map Σ and the measure $\widetilde{\nu}$ on $\widehat{G}$ diagonalising $\mathrm{ind}_H^G(\sigma)$ as in section 2.3.

If G is generic and H is compact, one can quite easily prove that $\mathrm{ind}_H^G(\sigma)$ is contained in the regular representation of G, thus showing that the diagonalisation of $\mathrm{ind}_H^G(\sigma)$ follows from Plancherel theory applied to G (actually, in §3.4 we followed a different and quicker approach, but Theorem 3.4.2 follows from Theorem 3.2.1, and the latter is an application of Plancherel theory applied to G).

There are few other cases in which $\mathrm{ind}_H^G(\sigma)$ can be diagonalised by known methods. One of them has been worked out by Kirillov [44] in the case G is a nilpotent Lie group and H is a generic closed subgroup of G. But the practical application of the method of Kirillov is rather complicated, and does not yeld to a compact form for covariant POM's like the quite simple expressions we obtained in eqs. (2.11) and (3.7). Moreover, the only nilpotent Lie group of interest in physics is the Heisenberg group, and, as we have seen in §3.5.2, its covariant POM's can be classified using the theory exposed in chapter 3.

A method for diagonalising $\mathrm{ind}_H^G(\sigma)$ can also be elaborated when G is the semidirect product $N \times' H$, with N normal abelian factor. We do not enter into details, but we only remark that this method can be used to describe the localisation observables for a relativistic quantum particle. For some hints about these facts we refer to the work of Castrigiano [22], [23] and to the last part of the book of Varadarajan [52].

Finally, the results in chapters 4 and 5 give an answer to some questions raised in [11]. In particular, Proposition 5.2.2 shows that in order that a position observable E_ρ and a momentum observable F_ν are functionally coexistent, the probability measures ρ and ν must form a Fourier couple in the sense of [11, sec. III.2.4]. Up to our knowledge, no example is known of a joint observable of a position and a momentum observable which is *not* a covariant phase space observable.

The last paragraph is of course intended according to our definition of position and momentum observables. We stress again that in the literature one often encounters many different approaches to the problem of position and momentum measurements in quantum mechanics, and that our results only refer to the covariant case.

Chapter 7

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