

INFORMATIONALLY COMPLETE JOINT MEASUREMENTS ON FINITE QUANTUM SYSTEMS

CLAUDIO CARMELI, TEIKO HEINOSAARI, AND ALESSANDRO TOIGO

ABSTRACT. We show that there are informationally complete joint measurements of two conjugated observables on a finite quantum system, meaning that they enable to identify all quantum states from their measurement outcome statistics. We further demonstrate that it is possible to implement a joint observable as a sequential measurement. If we require minimal noise in the joint measurement, then the joint observable is unique. If the dimension d is odd, then this observable is informationally complete. But if d is even, then the joint observable is not informationally complete and one has to allow more noise in order to obtain informational completeness.

1. INTRODUCTION

The general aim in quantum tomography is to identify quantum states from measurement outcome statistics. A collection of observables with this property is called *informationally complete* [1]. Even a single observable can be informationally complete, but then it must be a noncommutative positive operator valued measure (POVM) [2, 3]. We will study a class of informationally complete POVMs in dimension d with the minimal number of d^2 outcomes and we will explain how they can be implemented as sequential measurements of two d -outcome measurements. One can interpret the generated joint observable as a phase space measurement in the discrete phase space $\mathbb{Z}_d \times \mathbb{Z}_d$ [4, 5].

Claudio Carmeli, Dipartimento di Fisica, Università di Genova, Via Dodecaneso 33, I-16146 Genova, Italy
email: claudio.carmeli@gmail.com.

Teiko Heinosaari, Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku
email: teiko.heinosaari@utu.fi.

Alessandro Toigo, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy, and I.N.F.N., Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
email: alessandro.toigo@polimi.it.

There are some particularly interesting approaches to finite dimensional quantum tomography, and one of them is based on complete collections of mutually unbiased bases (MUBs) [6, 7, 8]. In a d -dimensional Hilbert space one needs $d + 1$ MUBs in order to be able to identify all quantum states, but it is not known if a complete set of MUBs exists in all dimensions. In fact, there is evidence that for $d = 6$ there is no complete set of MUBs [9, 10].

In our scheme we start from two mutually unbiased bases connected by the finite Fourier transform. They define a pair of complementary observables, which cannot be measured jointly. However, it is possible to realize their joint measurement if some additional noise is allowed. We show that their joint measurement can be chosen to be informationally complete, and that this can be realized as a sequential measurement where we first perform a ‘weak measurement’ in one basis and then another successive measurement in the other basis. Compared to the fact that one would need $d + 1$ complementary observables in order to reach informational completeness in separate measurements, it is remarkable that in the sequential scheme only two observables suffice.

The price to have a joint measurement is that the marginal observables are not the original complementary observables but their unsharp versions. We will analyze the required additional noise and characterize the optimal joint observable from this point of view. The qubit case has been first studied in [11], and our work generalizes those results to arbitrary finite dimension.

The covariant phase space observables, i.e., POVMs covariant under the finite Weyl-Heisenberg group, play a special role in our investigation. We prove that if a pair of conjugate observables have a joint measurement, then they also have a joint measurement which is a covariant phase space observable. Since every covariant phase space observable arises from a sequential measurement of two conjugate observables [12], the covariant phase space observables are an outstanding choice for finite dimensional quantum tomography.

The Weyl-Heisenberg group has also a pivotal role in the investigations of symmetric informationally complete (SIC) observables [13, 14]. It is generally believed that a Weyl-Heisenberg covariant SIC observable exists in every finite dimension and their existence is numerically tested in all dimensions up to 67 [15]. Our results show that any such observable has a neat sequential realization scheme.

There is an interesting difference between the even and odd dimensional Hilbert spaces. If we require minimal noise in both marginal observables, then their joint observable is unique. If d is odd, then this observable is informationally complete. But if d is even, then the joint

observable is not informationally complete. This result gives an additional aspect to the common observation that quantum tomography is different in even and odd dimensions [16].

2. PRELIMINARIES

In this section we fix some notations and introduce the basic concepts.

States and Observables. Let $\mathcal{H}$ be a finite dimensional Hilbert space, with $\dim \mathcal{H} = d \geq 2$. We denote by $\mathcal{L}(\mathcal{H})$ the vector space of all linear operators on $\mathcal{H}$. A positive operator $\varrho \in \mathcal{L}(\mathcal{H})$ having trace one is a *state*, and we denote by $\mathcal{S}(\mathcal{H})$ the set of all states.

Observables are generally described by *positive operator valued measures* (POVMs) [17, 18]. In this work we only consider observables with finite number of outcomes. Therefore, an observable can be defined as a function $A : x \mapsto A(x)$, where each $A(x)$ is a positive operator and $\sum_x A(x) = \mathbb{1}$. Here the sum runs over all $x \in \Omega_A$, where the set Ω_A is the collection of all possible measurement outcomes.

If a system is prepared in a state ϱ , then a measurement of an observable A will lead to an outcome x with the probability $\text{tr}[\varrho A(x)]$.

Informational completeness. An observable A is *informationally complete* if its measurement outcome probability distribution is sufficient to identify a unique state [1]. In other words, two different states must give rise to different probability distributions: for all pairs of states ϱ_1, ϱ_2 ,

$$\text{tr}[\varrho_1 A(j)] = \text{tr}[\varrho_2 A(j)] \quad \forall j \in \Omega_A \quad \Rightarrow \quad \varrho_1 = \varrho_2.$$

The informational completeness of an observable A is equivalent to the property that the linear span of the set $\{A(j) : j \in \Omega_A\}$ is $\mathcal{L}(\mathcal{H})$ [3, 19].

Joint measurability. Given two observables A and B , we say that they are *jointly measurable* if there exists a third observable C with $\Omega_C = \Omega_A \times \Omega_B$ and satisfying

$$(1) \quad \sum_{x \in \Omega_A} C(x, y) = B(y) \quad \forall y, \quad \sum_{y \in \Omega_B} C(x, y) = A(x) \quad \forall x.$$

In other words, A and B correspond to the ‘marginals’ of C . Any observable satisfying (1) is called a *joint observable* of A and B [20].

We recall that joint measurability is equivalent to the following [21]: there exists an observable G and stochastic matrices $[M_{xz}]$, $[M'_{yz}]$ such

that

$$(2) \quad \sum_z M_{xz} \mathbf{G}(z) = \mathbf{A}(x) \quad \forall x, \quad \sum_z M'_{yz} \mathbf{G}(z) = \mathbf{B}(y) \quad \forall y.$$

Hence, two observables are jointly measurable iff they can be ‘post-processed’ from a single observable.

We will use several times the following simple fact: if $\mathbf{C}$ and $\mathbf{C}'$ are joint observables of $\mathbf{A}$ and $\mathbf{B}$, then also all their convex combinations $t\mathbf{C} + (1-t)\mathbf{C}'$, $0 < t < 1$, are joint observables of $\mathbf{A}$ and $\mathbf{B}$. It follows that two jointly measurable observables have either a unique joint observable or infinitely many of them.

Another useful fact is related to unitary transformations. Let U be a unitary operator on $\mathcal{H}$. Two observables $\mathbf{A}$ and $\mathbf{B}$ are jointly measurable if and only if the observables UAU^* and UBU^* are jointly measurable. Indeed, it is easy to see that $\mathbf{C}$ is a joint observable of $\mathbf{A}$ and $\mathbf{B}$ if and only if UCU^* is a joint observable of UAU^* and UBU^* .

Instruments. An observable describes the statistics of the outcomes of a measurement but leaves open how the measurement disturbs the input state. In order to discuss this we need the concept of an *instrument* [22]. An instrument with finitely many outcomes is a mapping $\mathcal{I} : x \mapsto \mathcal{I}_x$ such that each $\mathcal{I}_x$ is a completely positive linear map on $\mathcal{L}(\mathcal{H})$ and $\sum_x \text{tr} [\mathcal{I}_x(\varrho)] = 1$ for all states ϱ .

The *adjoint map* $\mathcal{I}_x^*$ of $\mathcal{I}_x$ is defined via the usual trace duality

$$\text{tr} [S\mathcal{I}_x(T)] = \text{tr} [\mathcal{I}_x^*(S)T] \quad \forall S, T \in \mathcal{L}(\mathcal{H}).$$

In other words, $\mathcal{I}_x^*$ and $\mathcal{I}_x$ correspond to the Heisenberg and Schrödinger pictures, respectively.

Suppose that $\mathbf{A}$ is an observable. Then we say that an instrument $\mathcal{I}$ is *A-compatible* if $\mathcal{I}_x^*(\mathbb{1}) = \mathbf{A}(x)$ for every x . Every $\mathbf{A}$ -compatible instrument describes some particular kind of measurement of $\mathbf{A}$ [23].

An example of an $\mathbf{A}$ -compatible instrument is the *Lüders instrument* $\mathcal{I}^L$, defined by

$$\mathcal{I}_x^L(\varrho) = \sqrt{\mathbf{A}(x)}\varrho\sqrt{\mathbf{A}(x)}.$$

Any other $\mathbf{A}$ -compatible instrument $\mathcal{I}$ is of the form

$$\mathcal{I}_x(\varrho) = \mathcal{E}_x(\mathcal{I}_x^L(\varrho))$$

for some collection $\{\mathcal{E}_x\}$ of completely positive trace preserving maps on $\mathcal{L}(\mathcal{H})$ [24].

Sequential measurements. By a *sequential measurement* we mean a setting where two measurements are combined into a third measurement by performing them one after the other [25]. Generally, the order in which the measurements are performed is crucial [26].

Suppose we have two N -outcome observables A, B and we measure them subsequently; first A and then B . As a result, we have in total N^2 possible measurement outcomes. Generally, we do not obtain a joint measurement of A and B since the first measurement disturbs the input state. In fact, the overall measurement depends on the way we measure A . If the first measurement is described by an A -compatible instrument $\mathcal{I}$, then the overall observable C is given by

$$\mathrm{tr} [\varrho C(j, k)] = \mathrm{tr} [B(k) \mathcal{I}_j(\varrho)]$$

for all input states ϱ , or equivalently,

$$C(j, k) = \mathcal{I}_j^*(B(k)).$$

Let us notice that first marginal of C is always A , while the second marginal is a perturbed version of B and depends on the instrument $\mathcal{I}$.

3. EXAMPLE: SEQUENTIAL MEASUREMENTS OF σ_x AND σ_y

We start with a preliminary example, which is mainly a collection of well known facts. It hints the forthcoming developments and clarifies the aims of the later sections. We refer to [27] for more details and further references.

Fix $\mathcal{H} = \mathbb{C}^2$, and let A and B be the two observables corresponding to the measurements of spin- $\frac{1}{2}$ components in the directions x and y , respectively. Thus,

$$A(\pm 1) = \frac{1}{2}(\mathbb{1} \pm \sigma_x), \quad B(\pm 1) = \frac{1}{2}(\mathbb{1} \pm \sigma_y),$$

where σ_x, σ_y are the Pauli spin matrices.

Since A and B consist of projections and they do not mutually commute, it is not possible to measure them jointly. Moreover, if we measure them separately on two similarly prepared ensembles, we still cannot infer the unknown state.

An alternative way is to perform a sequential measurement. The first measurement has to be a weak measurement, meaning that we do not measure A but its unsharp version. We define an unsharp version A_λ of A by

$$A_\lambda(j) := \lambda A(j) + (1 - \lambda) \frac{1}{2} \mathbb{1}, \quad j = \pm 1.$$

Here $\lambda \in [0, 1]$ is a parameter quantifying the noise or imprecision. We can write A_λ in the form

$$A_\lambda(\pm 1) = \frac{1}{2}(\mathbb{1} \pm \lambda \sigma_x).$$

In a similar way we define an unsharp version B_γ of B by

$$B_\gamma(\pm 1) = \frac{1}{2}(\mathbb{1} \pm \gamma \sigma_y).$$

We want to study the disturbance of the first measurement on the system, and for this reason we define an instrument related to A_λ . A class of A_λ -compatible instruments can be defined by

$$\mathcal{J}_{\pm 1}(\varrho) = L_{\pm 1} \sqrt{A_\lambda(\pm 1)} \varrho \sqrt{A_\lambda(\pm 1)} L_{\pm 1}^*,$$

where L_1, L_{-1} are arbitrary unitary operators. If the subsequent measurement is a B -measurement, then the overall statistics of the sequential measurement is given by the observable

$$C(j, k) = \mathcal{J}_j^*(B(k)) = \sqrt{A_\lambda(j)} L_j^* B(k) L_j \sqrt{A_\lambda(j)}, \quad j, k = \pm 1.$$

The properties of C obviously depend on L_1 and L_{-1} . In the following we consider two different choices of $L_{\pm 1}$.

Optimal joint measurement. If we choose $L_{\pm 1} = \mathbb{1}$, then we obtain

$$C(j, k) = \frac{1}{4}(\mathbb{1} + j \lambda \sigma_x + k \sqrt{1 - \lambda^2} \sigma_y), \quad j, k = \pm 1.$$

In particular, the marginals are

$$\begin{aligned} C(j, +1) + C(j, -1) &= A_\lambda(j), \\ C(+1, k) + C(-1, k) &= \frac{1}{2}(\mathbb{1} + k \sqrt{1 - \lambda^2} \sigma_y) = B_{\sqrt{1 - \lambda^2}}(k). \end{aligned}$$

The joint observable C is an optimal approximate joint measurement of σ_x and σ_y . This means that the unsharp parameters λ and $\gamma = \sqrt{1 - \lambda^2}$ saturate the inequality

$$(3) \quad \lambda^2 + \gamma^2 \leq 1.$$

Indeed, it is known that this inequality is a necessary and sufficient criterion for two observables A_λ and B_γ to be jointly measurable [11]. Let us also notice that the joint observable of A_λ and B_γ is unique if $\lambda^2 + \gamma^2 = 1$ [11].

Informationally complete joint measurement. Another interesting option is to choose

$$L_{\pm 1} = \cos \frac{\theta}{2} \mathbb{1} \mp i \sin \frac{\theta}{2} \sigma_x$$

for some fixed angle $0 < \theta < \pi/2$. In this case we obtain

$$\mathbf{C}(j, k) = \frac{1}{4} (\mathbb{1} + j \lambda \sigma_x + k \cos \theta \sqrt{1 - \lambda^2} \sigma_y + j k \sin \theta \sqrt{1 - \lambda^2} \sigma_z)$$

and the marginals are

$$\mathbf{C}(j, +1) + \mathbf{C}(j, -1) = \mathbf{A}_\lambda(j),$$

$$\mathbf{C}(+1, k) + \mathbf{C}(-1, k) = \frac{1}{2} (\mathbb{1} + k \cos \theta \sqrt{1 - \lambda^2} \sigma_y) = \mathbf{B}_{\cos \theta \sqrt{1 - \lambda^2}}(k).$$

It is easy to see that the linear span of the four operators $\mathbf{C}(j, k)$, $j, k = \pm 1$, is the set of all 2×2 - complex matrices. It follows that the joint observable $\mathbf{C}$ is informationally complete.

The unsharpness parameters λ and $\gamma = \cos \theta \sqrt{1 - \lambda^2}$ do not saturate the inequality (3). Altering the parameter θ we can make the sum $\lambda^2 + \gamma^2$ as close to 1 as we want, hence we conclude that $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ admit an informationally complete joint observable if and only if

$$(4) \quad \lambda^2 + \gamma^2 < 1.$$

Finally, we remark that with the choices $\lambda = 1/\sqrt{3}$ and $\theta = \pi/4$ the joint observable $\mathbf{C}$ is a symmetric informationally complete (SIC) observable.

4. CONJUGATE OBSERVABLES

4.1. Mutually unbiased bases and complementary observables.

We start by recalling the usual definition of complementary observables in a finite d -dimensional Hilbert space and some related basic facts [28], [29]. We denote $\mathbb{Z}_d \equiv \{0, \dots, d-1\}$. Let $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$ be *mutually unbiased bases* (MUBs), i.e., they are orthonormal bases in $\mathcal{H}$ and

$$(5) \quad |\langle \varphi_j | \psi_k \rangle|^2 = 1/d \quad \forall j, k \in \mathbb{Z}_d.$$

We define two d -outcome observables $\mathbf{A}$ and $\mathbf{B}$ corresponding to $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$, respectively. Hence,

$$(6) \quad \mathbf{A}(j) = |\varphi_j\rangle\langle\varphi_j|, \quad \mathbf{B}(k) = |\psi_k\rangle\langle\psi_k|.$$

Obviously, two orthonormal bases $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\varphi'_j\}_{j \in \mathbb{Z}_d}$ define the same observable $\mathbf{A}$ iff $\varphi'_j = \alpha_j \varphi_j$ for some complex numbers α_j of modulus one.

The mutual unbiasedness condition (5) can be rephrased by saying that $\mathbf{A}$ and $\mathbf{B}$ are *complementary* observables, meaning that in any state ϱ where the outcome of $\mathbf{A}$ is predictable, the $\mathbf{B}$ -distribution is uniform (and vice versa). This entails that the following implications are valid for any state ϱ and all outcomes $j, k \in \mathbb{Z}_d$,

$$\begin{aligned} \text{tr} [\varrho \mathbf{A}(j)] = 1 &\Rightarrow \text{tr} [\varrho \mathbf{B}(k)] = 1/d \\ \text{tr} [\varrho \mathbf{B}(k)] = 1 &\Rightarrow \text{tr} [\varrho \mathbf{A}(j)] = 1/d. \end{aligned}$$

Since $\text{tr} [\varrho \mathbf{A}(j)] = 1$ iff $\varrho = |\varphi_j\rangle\langle\varphi_j|$, it is easy to see that the complementarity of $\mathbf{A}$ and $\mathbf{B}$ is indeed equivalent to the mutual unbiasedness of the bases $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$.

There is a canonical way to produce two mutually unbiased bases. In the following, suppose an orthonormal basis $\{\varphi_k\}_{k \in \mathbb{Z}_d}$ of $\mathcal{H}$ is fixed. Denoting $\omega \equiv e^{2\pi i/d}$, we define the following unitary representations U and V of the cyclic group $\mathbb{Z}_d$ in $\mathcal{H}$:

$$\begin{aligned} U_x \varphi_k &:= \varphi_{k+x} \\ V_y \varphi_k &:= \omega^{yk} \varphi_k \end{aligned}$$

for all $x, y, k \in \mathbb{Z}_d$. In the above formulas and in the rest of the paper, addition and multiplication of elements in $\mathbb{Z}_d$ are understood modulo d . (For instance, we will often use $-j = d - j$). It is easy to verify that

$$(7) \quad V_y U_x = \omega^{xy} U_x V_y \quad \forall x, y \in \mathbb{Z}_d.$$

The Fourier transform (with respect to the basis $\{\varphi_k\}_{k \in \mathbb{Z}_d}$) is the unitary operator $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$(8) \quad \mathcal{F} \varphi_k := \frac{1}{\sqrt{d}} \sum_{h \in \mathbb{Z}_d} \omega^{-hk} \varphi_h.$$

The adjoint operator $\mathcal{F}^*$ of $\mathcal{F}$ is given by

$$(9) \quad \mathcal{F}^* \varphi_k = \frac{1}{\sqrt{d}} \sum_{h \in \mathbb{Z}_d} \omega^{hk} \varphi_h = \mathcal{F} \varphi_{-k},$$

and we have $\mathcal{F}^2 \varphi_k = \mathcal{F}^{*2} \varphi_k = \varphi_{-k}$. We denote

$$\psi_k \equiv \mathcal{F}^* \varphi_k = \mathcal{F} \varphi_{-k},$$

and it is immediate to check that $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$ are MUBs, with $\langle \varphi_j | \psi_k \rangle = (1/\sqrt{d}) \omega^{jk}$.

The Fourier transform has the intertwining properties

$$\mathcal{F} U_x = V_x^* \mathcal{F}, \quad \mathcal{F} V_y = U_y \mathcal{F},$$

from which it follows that

$$\begin{aligned} U_x \psi_k &= \omega^{-xk} \psi_k \\ V_y \psi_k &= \psi_{k+y} . \end{aligned}$$

The observables $\mathbf{A}$ and $\mathbf{B}$ related to $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$, respectively, satisfy the following conditions for all $j, k, x, y \in \mathbb{Z}_d$:

$$(10) \quad U_x \mathbf{A}(j) U_x^* = \mathbf{A}(j+x), \quad V_y \mathbf{A}(j) V_y^* = \mathbf{A}(j)$$

and

$$(11) \quad U_x \mathbf{B}(k) U_x^* = \mathbf{B}(k), \quad V_y \mathbf{B}(k) V_y^* = \mathbf{B}(k+y) .$$

In other words, $\mathbf{A}$ is U -covariant and V -invariant, while $\mathbf{B}$ is U -invariant and V -covariant. We also note that $\mathbf{A}$ and $\mathbf{B}$ are conjugated by $\mathcal{F}$, i.e.,

$$(12) \quad \mathbf{B}(k) = \mathcal{F}^* \mathbf{A}(k) \mathcal{F}$$

for all $k \in \mathbb{Z}_d$. It is customary to say that $\mathbf{A}$ and $\mathbf{B}$ are *canonically conjugated* observables.

The conditions (10) – (11) are analogous to the symmetry properties of the usual position and momentum observables on the real line $\mathbb{R}$ (see e.g. [30]). In some situations the covariance properties may have some physical meaning or motivation. However, for our purposes they are just useful features that can be utilized later in our calculations.

4.2. Unsharp observables. Measurements of two complementary observables are incompatible and therefore have to be performed separately. This means that their measurements require different settings. However, it is possible to perform a simultaneous measurement of two complementary observables if we allow some additional imprecision or noise. In other words, we can measure jointly *unsharp versions* of $\mathbf{A}$ and $\mathbf{B}$.

We define an unsharp version $\mathbf{A}_\lambda$ of $\mathbf{A}$ by

$$\mathbf{A}_\lambda(j) := \lambda \mathbf{A}(j) + (1 - \lambda) \frac{1}{d} \mathbb{1} .$$

Here $\lambda \in [0, 1]$ is a parameter quantifying the noise. This type of noise is equivalent to the situation where an input state ϱ is first depolarized into a state $\lambda \varrho + (1 - \lambda)/d \mathbb{1}$ and then a measurement of $\mathbf{A}$ is performed.

More generally, if Λ is a probability distribution on $\mathbb{Z}_d$, then we define an unsharp version $\mathbf{A}_\Lambda$ of $\mathbf{A}$ by

$$(13) \quad \mathbf{A}_\Lambda(j) := \sum_{i \in \mathbb{Z}_d} \Lambda(j - i) \mathbf{A}(i) .$$

The special case $\mathbf{A}_\Lambda = \mathbf{A}_\lambda$ corresponds to the probability distribution Λ defined as

$$\Lambda(0) = \lambda + (1 - \lambda)/d, \quad \Lambda(j) = (1 - \lambda)/d \quad \text{if } j \neq 0.$$

We can also write this probability distribution in the form

$$\Lambda(j) = \lambda \delta(j) + (1 - \lambda) \mu(j)$$

where δ is the point distribution at 0 and μ is the uniform distribution on $\mathbb{Z}_d$, i.e.,

$$(14) \quad \delta(j) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}, \quad \mu(j) = \frac{1}{d} \quad \forall j.$$

In a similar way a probability distribution Γ on $\mathbb{Z}_d$ defines an unsharp version $\mathbf{B}_\Gamma$ of $\mathbf{B}$ by

$$(15) \quad \mathbf{B}_\Gamma(k) := \sum_{i \in \mathbb{Z}_d} \Gamma(k - i) \mathbf{B}(i).$$

A special class is, again, characterized by noise parameters $\gamma \in [0, 1]$ and we denote

$$\mathbf{B}_\gamma(k) := \gamma \mathbf{B}(k) + (1 - \gamma) \frac{1}{d} \mathbb{1}.$$

Naturally, there are also other type of approximations of $\mathbf{A}$ and $\mathbf{B}$ than the previously defined $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$. The usefulness of $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$ is that they satisfy the same covariance and invariance relations than $\mathbf{A}$ and $\mathbf{B}$, respectively. Namely, the observables $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$ satisfy the following conditions:

$$(16) \quad U_x \mathbf{A}_\Lambda(j) U_x^* = \mathbf{A}_\Lambda(j + x), \quad V_y \mathbf{A}_\Lambda(j) V_y^* = \mathbf{A}_\Lambda(j)$$

and

$$(17) \quad U_x \mathbf{B}_\Gamma(k) U_x^* = \mathbf{B}_\Gamma(k), \quad V_y \mathbf{B}_\Gamma(k) V_y^* = \mathbf{B}_\Gamma(k + y).$$

Thus, $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$ are *conjugated observables* although they need not be complementary anymore [12]. As we will see, two observables $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$ can have a joint observable even if they do not commute.

Remark 1. Suppose that $\tilde{\mathbf{A}}$ is a d -outcome observable satisfying

$$(18) \quad U_x \tilde{\mathbf{A}}(j) U_x^* = \tilde{\mathbf{A}}(j + x), \quad V_y \tilde{\mathbf{A}}(j) V_y^* = \tilde{\mathbf{A}}(j)$$

for all $j, x, y \in \mathbb{Z}_d$. Then $\tilde{\mathbf{A}} = \mathbf{A}_\Lambda$ for some probability distribution Λ . Namely, it follows from the second condition in (18) that $\tilde{\mathbf{A}}$ commutes with $\mathbf{A}$ (since $\mathbf{A}(j) = (1/d) \sum_y \omega^{-jy} V_y$) and hence $\tilde{\mathbf{A}}(j) = \sum_k p_{j,k} \mathbf{A}(k)$ for some real numbers $0 \leq p_{j,k} \leq 1$. The first condition in (18) then implies that $p_{j,k} = p_{0,k-j}$.

5. JOINT MEASUREMENTS

5.1. Covariant observables. We recall that two observables A_Λ and B_Γ are jointly measurable if they have a joint observable, i.e., an observable C on $\mathbb{Z}_d \times \mathbb{Z}_d$ such that

$$(19) \quad \sum_{k \in \mathbb{Z}_d} C(j, k) = A_\Lambda(j), \quad \sum_{j \in \mathbb{Z}_d} C(j, k) = B_\Gamma(k)$$

for all $j, k \in \mathbb{Z}_d$. A special class of joint observables turns out to be crucial for our developments. We say that an observable C on $\mathbb{Z}_d \times \mathbb{Z}_d$ is a *covariant phase space observable* if

$$(20) \quad U_x V_y C(j, k) V_y^* U_x^* = C(j + x, k + y)$$

for all $j, k, x, y \in \mathbb{Z}_d$. The covariant phase space observables have a simple form [22]. Namely, if C is a covariant phase space observable, then there is unique operator $T \in \mathcal{S}(\mathcal{H})$ such that $C = C_T$, where we have denoted

$$C_T(j, k) := \frac{1}{d} U_j V_k T V_k^* U_j^*, \quad j, k \in \mathbb{Z}_d.$$

Also, each $T \in \mathcal{S}(\mathcal{H})$ defines a covariant phase space observable by this formula. The correspondence $T \leftrightarrow C_T$ is therefore one-to-one and the elements in $\mathcal{S}(\mathcal{H})$ parametrize the covariant phase space observables.

The marginals of a covariant phase space observable C_T are conjugated observables on $\mathbb{Z}_d$. Indeed, a direct calculation shows that C_T has marginals A_Λ and B_Γ , with

$$(21) \quad \Lambda(j) = \text{tr}[A(-j)T], \quad \Gamma(k) = \text{tr}[B(-k)T].$$

(This calculation can be found in [12]).

The essential role of covariant phase space observables in our discussion becomes clear in the following observation.

Proposition 1. *If A_Λ and B_Γ are jointly measurable, then they have a joint observable which is a covariant phase space observable.*

Proof. Suppose that C is a joint observable of A_Λ and B_Γ . For each $x, y \in \mathbb{Z}_d$, we define an observable $C_{x,y}$ by

$$(22) \quad C_{x,y}(j, k) := U_x^* V_y^* C(j + x, k + y) V_y U_x, \quad j, k \in \mathbb{Z}_d.$$

Using the covariance and invariance properties (16) – (17) it is straightforward to verify that $C_{x,y}$ is a joint observable of A_Λ and B_Γ .

We then define $\tilde{C}$ to be the uniform mixture of all $C_{x,y}$, i.e.,

$$(23) \quad \tilde{C}(j, k) := \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} C_{x,y}(j, k), \quad j, k \in \mathbb{Z}_d.$$

Since every $C_{x,y}$ is a joint observable of A_Λ and B_Γ , also $\tilde{C}$ is their joint observable. A direct calculation, using (7), shows that $\tilde{C}$ is a covariant phase space observable. $\square$

We conclude from Proposition 1 that two observables A_Λ and B_Γ are jointly measurable iff their related probability distributions Λ and Γ are of the form (21) for some $T \in \mathcal{S}(\mathcal{H})$. Equations (21) can also be rewritten in a slightly different form. Namely, observe that, if $T \in \mathcal{S}(\mathcal{H})$, then there exists a unit vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ such that $T = \text{tr}_2[|\phi\rangle\langle\phi|]$, where tr_2 is the partial trace with respect to the second factor. (A vector state giving T via the partial trace is often called a *purification* of T).

Conversely, if $\phi \in \mathcal{H} \otimes \mathcal{H}$ is a unit vector, then $T = \text{tr}_2[|\phi\rangle\langle\phi|]$ is a state. Inserting this form into (21) we obtain

$$(24) \quad \Lambda(j) = \langle \phi | (A(-j) \otimes \mathbb{1}) \phi \rangle, \quad \Gamma(k) = \langle \phi | (B(-k) \otimes \mathbb{1}) \phi \rangle.$$

Note that if a vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ satisfies the above two equations for some probability densities Λ and Γ , then the normalization $\|\phi\| = 1$ is automatic. We thus have the following characterization of jointly measurable observables.

Proposition 2. *Let Λ, Γ be probability densities on $\mathbb{Z}_d$. The following facts are equivalent:*

- (i) *The observables A_Λ and B_Γ are jointly measurable.*
- (ii) *There exists a state $T \in \mathcal{S}(\mathcal{H})$ such that the probability densities Λ and Γ satisfy (21).*
- (iii) *There exists a vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ such that the probability densities Λ and Γ satisfy (24).*

Let us note that any A_Λ is jointly measurable with some B_Γ . Namely, for each Λ we can define a state T_Λ as

$$(25) \quad T_\Lambda := \sum_{j \in \mathbb{Z}_d} \Lambda(-j) |\varphi_j\rangle\langle\varphi_j|.$$

Then $\Lambda(j) = \text{tr}[A(-j)T_\Lambda]$ and A_Λ is thus jointly measurable with B_Γ , where Γ is defined as $\Gamma(k) = \text{tr}[B(-k)T_\Lambda]$.

Proposition 2 can be seen as a trade-off relation between the probability distributions Λ and Γ that describe the deviations of A_Λ and B_Γ from A and B , respectively. For instance, if $\Lambda = \delta$, then necessarily $\Gamma = \mu$. Hence, we recover the fact that A is jointly measurable only with the trivial observable and no other B_Γ .

We end this subsection with some additional observations.

Remark 2. *If two observables A_Λ and B_Γ are jointly measurable, they can have several different covariant phase space observables as their joint observables.*

For instance, let $\{\zeta_i\}_{i \in \mathbb{Z}_d}$ be an orthonormal basis which is mutually unbiased with respect to both orthonormal bases $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$. Then, for each $i \in \mathbb{Z}_d$, we have

$$\langle \zeta_i | A(-j) \zeta_i \rangle = \langle \zeta_i | B(-k) \zeta_i \rangle = 1/d.$$

Therefore, the marginals of the covariant phase space observables $C_{|\zeta_i\rangle\langle\zeta_i|}$ are the same although $C_{|\zeta_i\rangle\langle\zeta_i|} \neq C_{|\zeta_{i'}\rangle\langle\zeta_{i'}|}$ whenever $i \neq i'$.

Remark 3. *If two observables A_Λ and B_Γ are jointly measurable, they can have a joint observable which is not a covariant phase space observable.*

For instance, let $p : \mathbb{Z}_d \times \mathbb{Z}_d \rightarrow [0, 1]$ be a bivariate probability distribution with uniform margins. Then the observable $C(j, k) := p(j, k)\mathbb{1}$ is a joint observable of A_0 and B_0 . It is clear that C is a covariant phase space observable only if p is a uniform distribution. However, a bivariate probability distribution with uniform margins need not be uniform. For instance, if we set

$$p(i, j) = \frac{1}{d^2} \left(1 - \sin \left(2\pi \frac{ij}{d} \right) \right), \quad i, j \in \mathbb{Z}_d,$$

then $\sum_i p(i, j) = \sum_j p(i, j) = 1/d$, but p is not uniform.

The existence of non-covariant joint observables is not limited to the trivial observables A_0 and B_0 . Namely, suppose that C is a joint observable of A_0 and B_0 and C' is a joint observable of A_λ and B_γ . Then the convex combination $tC' + (1-t)C$, $0 < t < 1$, is a joint observable of $A_{t\lambda}$ and $B_{t\gamma}$. It is easy to see that if C' is a covariant phase space observable but C is not, then their convex combination $tC' + (1-t)C$ cannot be a covariant phase space observable.

Remark 4. *Suppose that A_λ and B_γ have a unique joint observable C_T among the covariant phase space observables and that $T^2 = T$. Then C_T is a unique joint observable of A_λ and B_γ .*

To prove this claim, let C be a joint observable of A_λ and B_γ . We need to show that $C = C_T$. We define $C_{x,y}$ and $\tilde{C}$ as in (22) – (23). Since $\tilde{C}$ is by its construction a covariant phase space observable, we must have $\tilde{C} = C_T$ by the assumption on uniqueness. In particular, each operator $\tilde{C}(j, k) = (1/d) U_j V_k T V_k^* U_j^*$ is rank-1. Since $0 \leq C_{x,y}(0, 0) \leq d^2 \tilde{C}(0, 0) = dT$ by (23), it follows that there exists a real constant

$0 \leq c(x, y) \leq d$ such that $\mathbf{C}_{x,y}(0, 0) = c(x, y) T$, hence

$$\mathbf{C}(x, y) = c(x, y) U_x V_y T V_y^* U_x^*$$

by (22). Suppose $\lambda \neq 1$. Since $\mathbf{C}$ has $\mathbf{A}_\lambda$ as its first marginal, then

$$\sum_{y \in \mathbb{Z}_d} c(x, y) U_x V_y T V_y^* U_x^* = \lambda \mathbf{A}(x) + (1 - \lambda) \frac{1}{d} \mathbb{1} \quad \forall x \in \mathbb{Z}_d.$$

The right hand side of this equation is a rank- d operator, while on the left hand side we have the sum of d operators with rank-1. It then follows that the set $\{U_x V_y T V_y^* U_x^*\}_{y \in \mathbb{Z}_d}$ is linearly independent in $\mathcal{L}(\mathcal{H})$. Since $\mathbf{C}$ and $\tilde{\mathbf{C}}$ have the same marginals,

$$\sum_{y \in \mathbb{Z}_d} c(x, y) U_x V_y T V_y^* U_x^* = \frac{1}{d} \sum_{y \in \mathbb{Z}_d} U_x V_y T V_y^* U_x^*$$

for all x , hence $c(x, y) = 1/d$ for all x, y by linear independence. The case $\lambda = 1$ is treated in a similar way, by taking the marginal $\mathbf{B}_\gamma$ in the place of $\mathbf{A}_\lambda$ (and now necessarily $\gamma \neq 1$). Therefore, $\mathbf{C}_T$ is the unique joint observable of $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$.

5.2. Unsharpness inequality. In this subsection we apply Proposition 2 to the cases where $\mathbf{A}_\Lambda = \mathbf{A}_\lambda$ and $\mathbf{B}_\Gamma = \mathbf{B}_\gamma$. These special type of marginal observables are interesting as we can quantify their unsharpnesses by single numbers λ and γ . In particular, we can ask how small λ and γ must be in order for $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ to become jointly measurable. We first notice that this question is, indeed, meaningful.

Proposition 3. *Let $\lambda, \gamma \in (0, 1]$. The following conditions are equivalent:*

- (i) $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ are jointly measurable.
- (ii) $\mathbf{A}_{\lambda'}$ and $\mathbf{B}_{\gamma'}$ are jointly measurable for all $0 \leq \lambda' \leq \lambda$ and $0 \leq \gamma' \leq \gamma$.
- (iii) $\mathbf{A}_{\lambda'}$ and $\mathbf{B}_{\gamma'}$ are jointly measurable for all $0 \leq \lambda' < \lambda$ and $0 \leq \gamma' < \gamma$.

Proof. Suppose that (i) holds and $0 < \gamma' < \gamma$. We denote $t := \gamma'/\gamma$ and hence $0 < t < 1$. We have

$$\mathbf{B}_{\gamma'}(k) = t \mathbf{B}_\gamma(k) + (1 - t) \frac{1}{d} \mathbb{1},$$

meaning that $\mathbf{B}_{\gamma'}$ is a convex combination of $\mathbf{B}_\gamma$ and the trivial observable $\mathbf{B}_0$. By the assumption $\mathbf{A}_\lambda$ is jointly measurable with $\mathbf{B}_\gamma$ and $\mathbf{A}_\lambda$ is also jointly measurable with the trivial observable $\mathbf{B}_0$ (since they commute). If $\mathbf{C}_1$ is a joint observable of $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ and $\mathbf{C}_2$ is a joint observable of $\mathbf{A}_\lambda$ and $\mathbf{B}_0$, then the convex combination $t \mathbf{C}_1 + (1 - t) \mathbf{C}_2$

is a joint observable of A_λ and $B_{\gamma'}$. Therefore, A_λ and $B_{\gamma'}$ are jointly measurable. We can interchange the roles of A_λ and B_γ and run the same argument, hence we obtain (ii).

It is clear that (ii) implies (iii). Hence, to complete the proof we need to show that (iii) implies (i).

Suppose that (iii) holds. We choose sequences (λ_n) and (γ_n) such that $0 < \lambda_n < \lambda$, $0 < \gamma_n < \gamma$ and $\lim_n \lambda_n = \lambda$, $\lim_n \gamma_n = \gamma$. For each n , we fix a state T_n such that the corresponding covariant phase space observable C_{T_n} is a joint observable of A_{λ_n} and B_{γ_n} . The set of states $\mathcal{S}(\mathcal{H})$ is compact in the operator norm topology, hence the sequence $(T_n)_n$ has a convergent subsequence. We denote by T the limit of this convergent subsequence. Using (21) we see that the covariant phase space observable C_T is a joint observable of A_λ and B_γ . Thus, (i) holds. $\square$

We conclude from Proposition 3 that for every $\lambda \in [0, 1]$, there is a number $\gamma_{\max}(\lambda) \geq 0$ such that A_λ and B_γ are jointly measurable iff $0 \leq \gamma \leq \gamma_{\max}(\lambda)$. Similarly, for every $\gamma \in [0, 1]$, there is a number $\lambda_{\max}(\gamma) \geq 0$ such that A_λ and B_γ are jointly measurable iff $0 \leq \lambda \leq \lambda_{\max}(\gamma)$.

We also know that $\gamma_{\max}(0) = \lambda_{\max}(0) = 1$ (since a trivial observable is jointly measurable with any other observable) and that $\gamma_{\max}(1) = \lambda_{\max}(1) = 0$ (see the discussion after Proposition 2).

Proposition 4. *The equality $\lambda_{\max}(x) = \gamma_{\max}(x)$ holds for all $x \in [0, 1]$.*

Proof. It is enough to show that A_λ and B_γ are jointly measurable if and only if A_γ and B_λ are such. Joint measurability of A_λ and B_γ means that there exists an observable C on $\mathbb{Z}_d \times \mathbb{Z}_d$ having marginals A_λ and B_γ , respectively. We set $\widehat{C}(j, k) := \mathcal{F}^* C(k, -j) \mathcal{F}$ for every $j, k \in \mathbb{Z}_d$. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}_d} \widehat{C}(j, k) &= \sum_{k \in \mathbb{Z}_d} \mathcal{F}^* C(k, -j) \mathcal{F} = \mathcal{F}^* B_\gamma(-j) \mathcal{F} \\ &= \mathcal{F}^{*2} A_\gamma(-j) \mathcal{F}^2 = A_\gamma(j), \\ \sum_{j \in \mathbb{Z}_d} \widehat{C}(j, k) &= \sum_{j \in \mathbb{Z}_d} \mathcal{F}^* C(k, -j) \mathcal{F} = \mathcal{F}^* A_\lambda(k) \mathcal{F} \\ &= B_\lambda(k). \end{aligned}$$

We conclude that $\widehat{C}$ is a joint observable of A_γ and B_λ , hence the latter two are jointly measurable. $\square$

We will now find out the function $\gamma_{\max}(\lambda)$, or, equivalently, $\lambda_{\max}(\gamma)$. Suppose that $\lambda, \gamma \in [0, 1]$ are such that $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ are jointly measurable. By Proposition 2 this is equivalent to the existence of a vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ satisfying

$$(26) \quad \langle \phi | (\mathbf{A}(j) \otimes \mathbb{1}) \phi \rangle = \lambda \delta(j) + (1 - \lambda) \mu(j),$$

$$(27) \quad \langle \phi | (\mathbf{B}(k) \otimes \mathbb{1}) \phi \rangle = \gamma \delta(k) + (1 - \gamma) \mu(k),$$

for all $j, k \in \mathbb{Z}_d$ (see (14) for the definition of δ and μ). We now give a condition on the parameters λ, γ which is necessary and sufficient for the existence of a vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ satisfying the above two equations. Moreover, we show that, for the extreme values of λ, γ , the vector ϕ is essentially unique.

Lemma 1. *Let $\lambda, \gamma \in [0, 1]$. Suppose there exists a vector $\phi \in \mathcal{H} \otimes \mathcal{H}$ satisfying (26) – (27). Then*

$$(28) \quad \gamma \leq \frac{1}{d} \left[(d-2)(1-\lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1} \right].$$

For any choice of a unit vector $\eta \in \mathcal{H}$, the vector

$$(29) \quad \phi = (\alpha_\lambda \varphi_0 + \beta_\lambda \psi_0) \otimes \eta,$$

with

$$\alpha_\lambda = \frac{1}{\sqrt{d}} \left[\sqrt{(d-1)\lambda + 1} - \sqrt{1-\lambda} \right], \quad \beta_\lambda = \sqrt{1-\lambda},$$

satisfies (26) – (27) with equality in (28). Hence, the right hand side in (28) is equal to $\gamma_{\max}(\lambda)$.

If ϕ' is a vector satisfying (26) – (27) with $\gamma = \gamma_{\max}(\lambda)$, then $\phi' = (\alpha_\lambda \varphi_0 + \beta_\lambda \psi_0) \otimes \eta'$ for some unit vector $\eta' \in \mathcal{H}$.

Proof. Suppose $\phi \in \mathcal{H} \otimes \mathcal{H}$ satisfies (26) – (27). We write ϕ in the form

$$\phi = \sum_{i \in \mathbb{Z}_d} \varphi_i \otimes \xi_i,$$

where $\{\xi_i\}_{i \in \mathbb{Z}_d}$ are vectors in $\mathcal{H}$. From (26) it follows that

$$\|\xi_i\|^2 = \lambda \delta(i) + (1 - \lambda) \mu(i),$$

hence there exist unit vectors $\{\eta_i\}_{i \in \mathbb{Z}_d}$ such that

$$\xi_0 = \sqrt{\frac{(d-1)\lambda + 1}{d}} \eta_0, \quad \xi_i = \sqrt{\frac{1-\lambda}{d}} \eta_i \quad \forall i \neq 0.$$

On the other hand, we have

$$\begin{aligned} \langle \phi | (\mathbf{B}(k) \otimes \mathbb{1}) \phi \rangle &= \sum_{i,j \in \mathbb{Z}_d} \langle \varphi_j \otimes \xi_j | (|\psi_k\rangle\langle\psi_k| \otimes \mathbb{1}) \varphi_i \otimes \xi_i \rangle \\ &= \frac{1}{d} \sum_{i,j \in \mathbb{Z}_d} \omega^{jk} \omega^{-ik} \langle \xi_j | \xi_i \rangle, \end{aligned}$$

so, by (27), we must have

$$\frac{1}{d} \sum_{i,j \in \mathbb{Z}_d} \omega^{jk} \omega^{-ik} \langle \xi_j | \xi_i \rangle = \gamma \delta(k) + (1 - \gamma) \mu(k).$$

This equation, evaluated at $k = 0$, gives

$$\begin{aligned} \left(1 - \frac{1}{d}\right) \gamma + \frac{1}{d} &= \frac{1}{d} \left\| \sum_{i \in \mathbb{Z}_d} \xi_i \right\|^2 \\ &= \frac{1}{d^2} \left\| \sqrt{(d-1)\lambda + 1} \eta_0 + \sum_{i \in \mathbb{Z}_d, i \neq 0} \sqrt{1 - \lambda} \eta_i \right\|^2. \end{aligned}$$

The maximum value of γ is then achieved when the right hand side of this equation is maximal, i.e., when there exists a unit vector $\eta \in \mathcal{H}$ such that $\eta_i = \eta \forall i \in \mathbb{Z}_d$. The corresponding maximum value $\gamma_{\max}$ of γ is given by

$$\left(1 - \frac{1}{d}\right) \gamma_{\max} + \frac{1}{d} = \frac{1}{d^2} \left(\sqrt{(d-1)\lambda + 1} + (d-1)\sqrt{1 - \lambda} \right)^2,$$

i.e.,

$$\gamma_{\max} = \frac{1}{d} \left[(d-2)(1 - \lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1} \right].$$

In order to show that, if the sequence $\{\xi_i\}_{i \in \mathbb{Z}_d}$ is chosen as above, then the corresponding vector

$$\begin{aligned} \phi &= \left(\sqrt{\frac{(d-1)\lambda + 1}{d}} \varphi_0 + \sqrt{\frac{1 - \lambda}{d}} \sum_{i \in \mathbb{Z}_d, i \neq 0} \varphi_i \right) \otimes \eta \\ &= (\alpha_\lambda \varphi_0 + \beta_\lambda \psi_0) \otimes \eta \end{aligned}$$

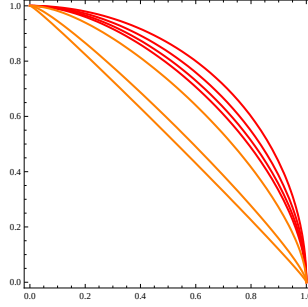


FIGURE 1. The boundary curve $\lambda \mapsto \gamma_{\max}(\lambda)$ for $d = 2, 3, 4, 5$ (red color) and for $d = 10, 100, 1000$ (orange color).

satisfies also (27) with $\gamma = \gamma_{\max}$ (and thus the maximum is indeed achieved by ϕ), we evaluate

$$\begin{aligned}
 \langle \phi | (\mathbf{B}(k) \otimes \mathbb{1}) \phi \rangle &= \langle \phi | (|\psi_k\rangle\langle\psi_k| \otimes \mathbb{1}) \phi \rangle \\
 &= |\langle \psi_k | \alpha_\lambda \varphi_0 + \beta_\lambda \psi_0 \rangle|^2 \\
 &= \left(\frac{\alpha_\lambda}{\sqrt{d}} + \beta_\lambda \delta(k) \right)^2 \\
 &= \frac{\alpha_\lambda^2}{d} + \left(\beta_\lambda^2 + 2 \frac{\alpha_\lambda \beta_\lambda}{\sqrt{d}} \right) \delta(k) \\
 &= (1 - \gamma_{\max}) \mu(k) + \gamma_{\max} \delta(k),
 \end{aligned}$$

which is (27). $\square$

As a consequence of the above discussion, we obtain an inequality for the unsharpnesses of two jointly measurable observables $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$.

Proposition 5. *Two observables $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ are jointly measurable if and only if*

$$(30) \quad \gamma \leq \gamma_{\max}(\lambda) = \frac{1}{d} \left[(d-2)(1-\lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1} \right],$$

(or, equivalently, its modified form under the exchange $\gamma \leftrightarrow \lambda$).

If $\gamma = \gamma_{\max}(\lambda)$, then $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ have a unique joint observable. This unique joint observable is the covariant phase space observable $\mathbf{C}_T$ defined by the state

$$T = |\chi_\lambda\rangle\langle\chi_\lambda|, \quad \chi_\lambda = \alpha_\lambda \varphi_0 + \beta_\lambda \psi_0,$$

with

$$(31) \quad \alpha_\lambda = \frac{1}{\sqrt{d}} \left[\sqrt{(d-1)\lambda + 1} - \sqrt{1-\lambda} \right], \quad \beta_\lambda = \sqrt{1-\lambda}.$$

Proof. If A_λ and B_γ are jointly measurable, then the inequality follows from Proposition 2 and Lemma 1. Conversely, if $\gamma_{\max}$ is given by the right hand side of (30), then the pair A_λ and $B_{\gamma_{\max}}$ are jointly measurable again by an application of Proposition 2 and Lemma 1. Then, A_λ and B_γ are jointly measurable by Proposition 3.

Now suppose λ and γ achieve the bound (30), and let C_T be a covariant joint observable of A_λ and B_γ . Pick $\phi \in \mathcal{H} \otimes \mathcal{H}$ such that $T = \text{tr}_2[|\phi\rangle\langle\phi|]$. As ϕ satisfies (26) – (27) with $\gamma = \gamma_{\max}$, by Lemma 1 it must be given by (29) for some choice of a unit vector $\eta \in \mathcal{H}$, and $T = \text{tr}_2[|\phi\rangle\langle\phi|] = |\chi_\lambda\rangle\langle\chi_\lambda|$, with χ_λ as in the statement of the proposition.

Finally, we need to prove that C_T is the unique joint observable (and not only unique among covariant phase space observables). We notice that $T^2 = T$, and the claim thus follows from Remark 4. $\square$

The graph of the function $\lambda \mapsto \gamma_{\max}(\lambda)$ is a part of an ellipse. In Fig. 1 we have depicted it for $d = 2, 3, 4, 5, 10, 100, 1000$.

Example 1. Suppose that two jointly measurable observables A_λ and B_γ are ‘equally unsharp’ but as close to A and B as possible, i.e.,

$$\gamma = \lambda = \lambda_{\max}(\gamma).$$

In this case Proposition 5 gives

$$(32) \quad \gamma = \lambda = \frac{d + \sqrt{d} - 2}{2(d-1)} = \frac{1}{2} \left(1 + \frac{1}{1 + \sqrt{d}} \right).$$

The observables A_λ and B_γ then have a unique joint observable, which is the covariant phase space observable C_T associated to the state $T = |\chi\rangle\langle\chi|$, with

$$\chi = \sqrt{\frac{\sqrt{d}}{2(1 + \sqrt{d})}} (\varphi_0 + \psi_0).$$

The vector state χ is hence an equal superposition of the vector states φ_0 and ψ_0 .

By a direct calculation one can verify that the inequality (30) can be rewritten in the following equivalent form which is symmetric in λ and γ .

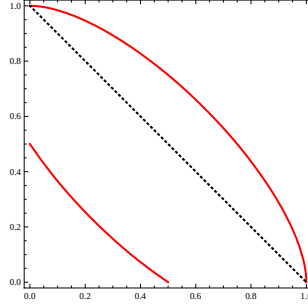


FIGURE 2. The ellipse (red) and the line (dashed) from (33) for $d = 8$. Only the upper side of the ellipse is relevant for the joint measurability.

Proposition 6. *Two observables A_λ and B_γ are jointly measurable iff*

$$(33) \quad \text{either } \gamma + \lambda \leq 1 \quad \text{or} \quad \gamma^2 + \lambda^2 + \frac{2(d-2)}{d}(1-\gamma)(1-\lambda) \leq 1.$$

The second inequality in (33) describes a full ellipse. Therefore, the first condition $\gamma + \lambda \leq 1$ is needed to ignore the lower part of the ellipse, which is not a correct boundary for joint measurability. This is depicted in Fig. 2.

Let us notice that for $d = 2$ the latter inequality in (33) becomes $\gamma^2 + \lambda^2 \leq 1$, and then the first condition is redundant. We thus recover the single condition stated in (3) and first proved in [11]. Also, for $d = 3$ and $d = 4$ a direct calculation shows that the first condition in (33) is redundant and the latter quadratic inequality is necessary and sufficient for the joint measurability. For $d \geq 5$ we need both conditions in (33).

By inspecting the function $\lambda \mapsto \gamma_{\max}(\lambda)$ we see that for every $\epsilon > 0$, there is a pair of observables A_λ and B_γ such that they are not jointly measurable and $\lambda + \gamma < 1 + \epsilon$. Thus, the criterion $\gamma + \lambda \leq 1$ is the best *sufficient* condition for joint measurability which is linear and symmetric in λ and γ .

The best linear and symmetric *necessary* condition for joint measurability is achieved by taking the tangent of the boundary curve in the point where it crosses the line $\gamma = \lambda$. In this way, we obtain the following conclusion.

Proposition 7. *If A_λ and B_γ are jointly measurable, then*

$$(34) \quad \gamma + \lambda \leq 1 + \frac{\sqrt{d} - 1}{d - 1}.$$

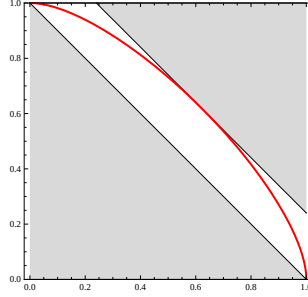


FIGURE 3. In this picture $d = 10$. The gray regions represent the necessary and sufficient linear conditions. In the white stripe one has to invoke the quadratic ellipse criterion, whereas otherwise the joint measurability can be deduced from the simple linear criteria.

One can also see Proposition 7 in the opposite order; if $\gamma + \lambda > 1 + \frac{\sqrt{d}-1}{d-1}$, then A_λ and B_γ are not jointly measurable. In Fig. 3 we have depicted the linear necessary and sufficient conditions in the case $d = 10$.

5.3. Non-covariant observables. So far, we have concentrated on covariant observables A_Λ and B_Γ . Let us have a short look on a class of non-covariant observables.

Let p and r be two probability distributions on $\mathbb{Z}_d$. We define

$$A_{\lambda;p}(j) := \lambda A(j) + (1 - \lambda)p(j)\mathbb{1}$$

and

$$B_{\gamma;r}(k) := \gamma B(k) + (1 - \gamma)r(k)\mathbb{1}.$$

It is straightforward to verify that $A_{\lambda;p}$ is U -covariant and V -invariant iff p is the uniform distribution on $\mathbb{Z}_d$, in which case $A_{\lambda;p} = A_\lambda$. (Analogous statement holds for $B_{\gamma;r}$). The following result is a generalization of Proposition 5 in [27].

Proposition 8. *If $A_{\lambda;p}$ and $B_{\gamma;r}$ are jointly measurable, then A_λ and B_γ are jointly measurable.*

Proof. Suppose that $A_{\lambda;p}$ and $B_{\gamma;r}$ are jointly measurable and let C be their joint observable. As in the proof of Proposition 1, we define the observable $\tilde{C}$, given by

$$\tilde{C}(j, k) := \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} U_x^* V_y^* C(j + x, k + y) V_y U_x, \quad j, k \in \mathbb{Z}_d.$$

For each j , we have

$$\begin{aligned}
\sum_{k \in \mathbb{Z}_d} \tilde{\mathbf{C}}(j, k) &= \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} U_x^* V_y^* \sum_{k \in \mathbb{Z}_d} \mathbf{C}(j+x, k+y) V_y U_x \\
&= \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} U_x^* V_y^* \mathbf{A}_{\lambda; p}(j+x) V_y U_x \\
&= \lambda \mathbf{A}(j) + (1-\lambda) \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} p(j+x) \mathbb{1} \\
&= \mathbf{A}_\lambda(j).
\end{aligned}$$

In a similar way we obtain $\sum_{j \in \mathbb{Z}_d} \tilde{\mathbf{C}}(j, k) = \mathbf{B}_\gamma(k)$ for every k . Therefore, $\tilde{\mathbf{C}}$ is a joint observable for $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$. $\square$

As a consequence of Propositions 5 and 8 we conclude the following necessary criterion for joint measurability.

Corollary 1. *If two observables $\mathbf{A}_{\lambda; p}$ and $\mathbf{B}_{\gamma; r}$ are jointly measurable, then*

$$(35) \quad \gamma \leq \frac{1}{d} \left[(d-2)(1-\lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1} \right].$$

A necessary and sufficient inequality for the joint measurability of $\mathbf{A}_{\lambda; p}$ and $\mathbf{B}_{\gamma; r}$ must contain also p and r in a form or in another. It is thus clear that Corollary 1 does not give a sufficient condition. A necessary and sufficient condition in the case $d = 2$ has been obtained in [31, 32, 33].

We remark that a general necessary condition for the joint measurability of two observables on a finite dimensional system has been presented in [34]. A comparison to Proposition 5 shows that this condition is not sufficient. We leave it as an open problem to find a necessary and sufficient condition for the joint measurability of $\mathbf{A}_{\lambda; p}$ and $\mathbf{B}_{\gamma; r}$.

6. INFORMATIONAL COMPLETENESS

We will now study the informational completeness of joint observables of $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$. Let us first recall that the informational completeness of a covariant phase space observable $\mathbf{C}_T$ is equivalent to the criterion

$$(36) \quad \text{tr}[T U_x V_y] \neq 0 \quad \forall x, y \in \mathbb{Z}_d.$$

This result has been discussed e.g. in [35, 36, 37]. For completeness, we provide a proof in Appendix.

Proposition 9. *Suppose A_λ and B_γ are two observables with $\lambda \notin \{0, 1\}$ and $\gamma = \gamma_{\max}(\lambda)$. Then A_λ and B_γ have a unique joint observable C . The observable C is informationally complete if and only if d is odd.*

Proof. From Proposition 5 we know A_λ and B_γ have a unique joint observable C_T , generated by the state $T = |\chi_\lambda\rangle\langle\chi_\lambda|$, with $\chi_\lambda = \alpha_\lambda\varphi_0 + \beta_\lambda\psi_0$. The informational completeness of C_T is equivalent to the condition (36), and a straightforward calculation gives

$$\text{tr}[TU_xV_y] = \langle\chi_\lambda|U_xV_y\chi_\lambda\rangle = \alpha_\lambda^2\delta_{x,0} + \beta_\lambda^2\delta_{y,0} + \frac{\alpha_\lambda\beta_\lambda}{\sqrt{d}}(\omega^{-xy} + 1).$$

Let us first notice that $\alpha_\lambda > 0$ and $\beta_\lambda > 0$ since both λ and γ are nonzero (see (31)). Hence, $\text{tr}[TU_xV_y] = 0$ exactly when $\omega^{-xy} = -1$. The latter condition is equivalent to $2xy \equiv d \pmod{2d}$. We conclude that the informational completeness of C_T is equivalent to the fact that the equation $2x = d \pmod{2d}$ has no solution $x \in \mathbb{Z}_d$, and this holds if and only if d is odd. $\square$

In Proposition 9 the crucial assumption is that $\gamma = \gamma_{\max}(\lambda)$. This guarantees that A_λ and B_γ have a unique joint observable. If we have $0 < \gamma < \gamma_{\max}(\lambda)$, then A_λ and B_γ have infinitely many joint observables. In this case it is always possible to choose an informationally complete joint observable, as we prove in the following.

Proposition 10. *Suppose A_λ and B_γ are two observables with $\lambda \notin \{0, 1\}$ and $0 < \gamma < \gamma_{\max}(\lambda)$. Then they have an informationally complete covariant joint observable.*

Proof. Let (γ_0, λ_0) be the intersection of the half line $\mathbb{R}_+(\gamma, \lambda)$ with the boundary of the domain (33) in $\mathbb{R}_+^2$, and let $t_0 > 1$ such that $t_0(\gamma, \lambda) = (\gamma_0, \lambda_0)$. Let $\tau = 1 - 1/t_0 \in (0, 1)$. We treat separately the cases of odd and even d .

1) Suppose that $d = 2n$ is even. For all $k \in \mathbb{Z}_d$, we denote

$$X_k := \frac{i}{2}(|\varphi_{-k}\rangle\langle\varphi_0| - |\varphi_0\rangle\langle\varphi_{-k}| + |\varphi_k\rangle\langle\varphi_0| - |\varphi_0\rangle\langle\varphi_k|).$$

The linear maps X_k are selfadjoint trace zero operators for every k , and it is easy to check that

$$\sum_{x \in \mathbb{Z}_d} U_x V_y X_k V_y^* U_x^* = 0, \quad \sum_{y \in \mathbb{Z}_d} U_x V_y X_k V_y^* U_x^* = 0.$$

We introduce the selfadjoint operators

$$X = \sum_{k \in \mathbb{Z}_d} X_k,$$

and, for $\kappa > 0$,

$$S_\kappa = \frac{1}{d}\mathbb{1} + \kappa X.$$

If $\kappa < 1/(d\|X\|)$, then $S_\kappa \in \mathcal{S}(\mathcal{H})$. Moreover, the associated covariant phase space observable $\mathbf{C}_{S_\kappa}$ has trivial marginals $\mathbf{A}_0$ and $\mathbf{B}_0$. A straightforward calculation gives

$$\text{tr}[S_\kappa U_x V_y] = \delta_{x,0}\delta_{y,0} + i\kappa(\omega^{-xy} - 1).$$

The covariant phase space observable $\mathbf{C}_{T_\kappa}$ associated to the state $T_\kappa = (1-\tau)|\chi_{\lambda_0}\rangle\langle\chi_{\lambda_0}| + \tau S_\kappa$, with $\chi_{\lambda_0} = \alpha_{\lambda_0}\varphi_0 + \beta_{\lambda_0}\psi_0$, has margins $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$. Moreover,

$$\begin{aligned} \text{tr}[T_\kappa U_x V_y] &= (1-\tau)(\alpha_{\lambda_0}^2\delta_{x,0} + \beta_{\lambda_0}^2\delta_{y,0}) + \tau\delta_{x,0}\delta_{y,0} \\ (37) \quad &+ (1-\tau)\frac{\alpha_{\lambda_0}\beta_{\lambda_0}}{\sqrt{d}}(\omega^{-xy} + 1) + i\kappa\tau(\omega^{-xy} - 1). \end{aligned}$$

Let

$$\varepsilon = \min_{\{k \in \mathbb{Z}_d, k \neq n\}} |\omega^k + 1|, \quad \delta = \max_{\{k \in \mathbb{Z}_d\}} |\omega^k + 1|.$$

For $\kappa < \min \left\{ \alpha_{\lambda_0}\beta_{\lambda_0}(1-\tau)\varepsilon/(\tau\delta\sqrt{d}), 1/(d\|X\|) \right\}$, the right hand side of (37) is nonzero for all $x, y \in \mathbb{Z}_d$, which proves informational completeness of $\mathbf{C}_{T_\kappa}$ by the criterion (36).

2) Suppose that d is odd. Then, for $T = (1-\tau)|\chi_{\lambda_0}\rangle\langle\chi_{\lambda_0}| + (\tau/d)\mathbb{1}$, the associated covariant phase space observable $\mathbf{C}_T$ has margins $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$, and

$$\text{tr}[T U_x V_y] = (1-\tau) \left[\alpha_{\lambda_0}^2\delta_{x,0} + \beta_{\lambda_0}^2\delta_{y,0} + \frac{\alpha_{\lambda_0}\beta_{\lambda_0}}{\sqrt{d}}(\omega^{-xy} + 1) \right] + \tau\delta_{x,0}\delta_{y,0},$$

which is nonzero for all $x, y \in \mathbb{Z}_d$. The informational completeness of $\mathbf{C}_T$ then follows from the criterion (36). $\square$

The two trivial cases $\lambda = 0$ or $\gamma = 0$ are not very interesting, but for completeness we make the following observation.

Proposition 11. *Suppose $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ are two observables with $\lambda = 0$ or $\gamma = 0$. Then they have no informationally complete joint observable.*

Proof. We consider only the case $\lambda = 0$, the case $\gamma = 0$ being similar. Suppose that $\mathbf{C}$ is a joint observable of $\mathbf{A}_0 = \mu\mathbb{1}$ and $\mathbf{B}_\lambda$. We have

$$\sum_{k \in \mathbb{Z}_d} \mathbf{C}(j, k) = \mathbf{A}_0(j) = \frac{1}{d}\mathbb{1} \quad \forall j \in \mathbb{Z}_d,$$

hence

$$\text{span} \{ \mathbf{C}(j, k) \mid j, k \in \mathbb{Z}_d \} = \text{span} \{ \mathbb{1}, \mathbf{C}(j, k) \mid j \in \mathbb{Z}_d, k \in \mathbb{Z}_d \setminus \{0\} \}$$

and then, for $d \geq 2$,

$$\dim \text{span} \{C(j, k) \mid j, k \in \mathbb{Z}_d\} \leq 1 + d(d-1) < d^2.$$

Thus, C is not informationally complete. $\square$

7. SEQUENTIAL IMPLEMENTATION OF JOINT OBSERVABLES

In this section we discuss the sequential implementation of joint observables of A_Λ and B_Γ in the light of the recent results obtained in [12] and [38]. For illustrative purposes, we point out that two naive methods do not work.

7.1. Nondisturbing measurement. Suppose that A_Λ and B_Γ are jointly measurable, i.e., they satisfy the condition stated in Proposition 2. The most uncomplicated way to realize their joint measurement would be to perform an A_Λ -measurement without disturbing the subsequent B_Γ -measurement. In terms of instruments, this would mean that we choose an A_Λ -compatible instrument $\mathcal{I}$ such that

$$\sum_j \mathcal{I}_j^*(B_\Gamma(k)) = B_\Gamma(k)$$

for all k . However, this type of measurement is typically not possible since a quantum measurement necessarily disturbs the input state.

Let us first notice that A_λ and B_γ commute if and only if $\lambda\gamma = 0$, meaning that one of them is a trivial observable. Generally, a non-disturbing measurement can be possible even if two observables do not commute. But applying Proposition 3 from [38] we see that this possibility is excluded whenever B_γ is informationally equivalent with B in the sense that the linear spans of the sets $\{B_\gamma(k) : k \in \mathbb{Z}_d\}$ and $\{B(k) : k \in \mathbb{Z}_d\}$ are equal. This property is satisfied by any B_γ with $\gamma \neq 0$. Therefore, whenever both observables are nontrivial, then A_λ -measurement disturbs the subsequent B_γ -measurement and the resulting observable is not a joint measurement of A_λ and B_γ .

7.2. Measuring only part of the ensemble. Suppose we have a measurement setup for A and that the corresponding instrument is $\mathcal{I}$. We can implement an unsharp observable A_λ by performing the A -measurement in a randomly chosen λ -part of the ensemble and doing nothing for the rest $(1 - \lambda)$ -part. The corresponding A_λ -compatible instrument $\mathcal{I}'$ is then

$$(38) \quad \mathcal{I}'_j(\varrho) = \lambda \mathcal{I}_j(\varrho) + \frac{1 - \lambda}{d} \varrho.$$

This is clearly a very direct way to decrease the disturbance that an A -measurement would cause. By measuring the observable B after the

first measurement, one could expect to have a useful joint measurement of $\mathbf{A}_\lambda$ and some approximate version of $\mathbf{B}$. However, this type of method does not yield an informationally complete joint measurement.

The observable $\mathbf{A}$ consists of rank-1 operators, and any $\mathbf{A}$ -compatible instrument $\mathcal{I}$ is of the form

$$(39) \quad \mathcal{I}_j(\varrho) = \text{tr}[\varrho \mathbf{A}(j)] \xi_j$$

for some set of states $\{\xi_j : j \in \mathbb{Z}_d\}$ [38]. If we insert this form into (38), we see that a sequential measurement consisting of $\mathcal{I}'$ followed by a $\mathbf{B}$ -measurement leads to the joint observable

$$\mathbf{C}(j, k) = \lambda \text{tr}[\xi_j \mathbf{B}(k)] \mathbf{A}(j) + \frac{1 - \lambda}{d} \mathbf{B}(k), \quad j, k \in \mathbb{Z}_d.$$

The linear span of the set $\{\mathbf{C}(j, k) : j, k \in \mathbb{Z}_d\}$ is contained in the linear span of the union $\{\mathbf{A}(j) : j \in \mathbb{Z}_d\} \cup \{\mathbf{B}(k) : k \in \mathbb{Z}_d\}$. The latter is strictly smaller than $\mathcal{L}(\mathcal{H})$, hence $\mathbf{C}$ is not informationally complete. We also see that this kind of approach cannot give more information than separate measurements of $\mathbf{A}$ and $\mathbf{B}$ would give.

7.3. General joint observables. We recall from [38] that every joint observable of $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$ can be implemented as a sequential measurement of $\mathbf{A}_\Lambda$ followed by a measurement of $\mathbf{B}$. Namely, suppose that $\mathbf{C}$ is a joint observable of $\mathbf{A}_\Lambda$ and $\mathbf{B}_\Gamma$. We define an instrument $\mathcal{I}$ by

$$(40) \quad \mathcal{I}_j(\varrho) = \sum_{k \in \mathbb{Z}_d} \text{tr}[\varrho \mathbf{C}(j, k)] \mathbf{B}(k).$$

This is an $\mathbf{A}_\Lambda$ -compatible instrument, and from $\mathbf{B}(k)\mathbf{B}(k') = \delta_{k,k'}\mathbf{B}(k)$ it follows that

$$\text{tr}[\mathbf{B}(k)\mathcal{I}_j(\varrho)] = \text{tr}[\varrho \mathbf{C}(j, k)].$$

Hence, $\mathbf{C}(j, k) = \mathcal{I}_j^*(\mathbf{B}(k))$, and we conclude that $\mathbf{C}$ is implemented as a sequential measurement of $\mathbf{A}_\Lambda$ followed by a measurement of $\mathbf{B}$, as claimed.

7.4. Covariant phase space observables. The instrument defined in (40) may look quite artificial and before we know the structure of $\mathbf{C}$, the formula does not give us any hint on the structure of $\mathcal{I}$. In contrast, every covariant phase space observable can be implemented as a sequential measurement of $\mathbf{A}_\Lambda$ and $\mathbf{B}$ in a very specific form.

As explained in [12], every covariant $\mathbf{A}_\Lambda$ -compatible instrument gives rise to a covariant phase space observable. Covariance of an instrument $\mathcal{I}$ here means that

$$(41) \quad U_x V_y \mathcal{I}_j(V_y^* U_x^* \varrho U_x V_y) V_y^* U_x^* = \mathcal{I}_{j+x}(\varrho)$$

for all $x, y, j \in \mathbb{Z}_d$ and $\varrho \in \mathcal{S}(\mathcal{H})$. It is straightforward to verify that the joint observable $\mathbf{C}(j, k) := \mathcal{I}_j^*(\mathbf{B}(k))$ is a covariant phase space observable.

We demonstrate this method by choosing the $\mathbf{A}_\lambda$ -compatible Lüders instrument $\mathcal{I}^L$, defined as

$$\mathcal{I}_j^L(\varrho) = \sqrt{\mathbf{A}_\lambda(j)} \varrho \sqrt{\mathbf{A}_\lambda(j)}.$$

It is straightforward to see that $\mathcal{I}^L$ satisfies (41). The covariant joint observable is then

$$\mathbf{C}(j, k) = \sqrt{\mathbf{A}_\lambda(j)} \mathbf{B}(k) \sqrt{\mathbf{A}_\lambda(j)},$$

and its associated state is

$$T = d\mathbf{C}(0, 0) = d\sqrt{\mathbf{A}_\lambda(0)}|\psi_0\rangle\langle\psi_0|\sqrt{\mathbf{A}_\lambda(0)}.$$

Since

$$\begin{aligned} \sqrt{\mathbf{A}_\lambda(j)} &= \sqrt{\frac{1-\lambda}{d}} \mathbb{1} + \left(\sqrt{\frac{(d-1)\lambda+1}{d}} - \sqrt{\frac{1-\lambda}{d}} \right) \mathbf{A}(j) \\ &= \frac{\beta_\lambda}{\sqrt{d}} \mathbb{1} + \alpha_\lambda |\varphi_j\rangle\langle\varphi_j| \end{aligned}$$

and

$$\sqrt{\mathbf{A}_\lambda(0)}\psi_0 = \frac{1}{\sqrt{d}} (\beta_\lambda\psi_0 + \alpha_\lambda\varphi_0) = \frac{1}{\sqrt{d}}\chi_\lambda,$$

we see that $T = |\chi_\lambda\rangle\langle\chi_\lambda|$, hence by Proposition 5 the marginal $\mathbf{B}_\gamma$ is such that $\gamma = \gamma_{\max}(\lambda)$.

In conclusion, this type of sequential measurement of $\mathbf{A}_\lambda$ and $\mathbf{B}$ is effectively a joint measurement of $\mathbf{A}_\lambda$ and $\mathbf{B}_\gamma$ with minimal unsharpnesses.

8. DISCUSSION

In our investigation we have concentrated on canonically conjugated pairs of observables, i.e., the orthonormal bases $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$ have been assumed to be Fourier connected with respect to the Fourier transform of the cyclic group $\mathbb{Z}_d$; see (8). Equivalently, we have assumed that the two bases satisfy $\langle\varphi_j|\psi_k\rangle = (1/\sqrt{d})\omega^{jk}$ for all $j, k \in \mathbb{Z}_d$. As a consequence the observables $\mathbf{A}$ and $\mathbf{B}$, both defined on $\Omega_{\mathbf{A}} = \Omega_{\mathbf{B}} = \mathbb{Z}_d$ as $\mathbf{A}(j) = |\varphi_j\rangle\langle\varphi_j|$ and $\mathbf{B}(k) = |\psi_k\rangle\langle\psi_k|$, satisfy the covariance and invariance conditions (10) – (11), which turn out to be very useful in our calculations.

Our approach covers more cases than it may seem at the first sight. Namely, we recall that two orthonormal bases $\{\varphi_j\}_j$ and $\{\varphi'_j\}_j$ define the same observable iff there are complex numbers α_j with $|\alpha_j| = 1$

such that $\varphi'_j = \alpha_j \varphi_j$. To illustrate an application of this many-to-one correspondence, suppose that the dimension d is an odd prime number, say $d = p$ (the generalization to the case $d = p^r$, with r positive integer, is straightforward). In this case it is easy to give a full set of $p+1$ MUBs [8]; fix an orthonormal basis $\{\varphi_j\}_{j \in \mathbb{Z}_p}$ and define p orthonormal bases $\{\psi_k^a\}_{k \in \mathbb{Z}_p}$, each one labeled by $a \in \mathbb{Z}_p$, by

$$\psi_k^a = \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{Z}_p} \omega^{ax^2 + kx} \varphi_x.$$

The fact that these are MUBs follows from the Gauss summation formula

$$(42) \quad \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{Z}_p} \omega^{ax^2} = \left(\frac{a}{p}\right) \times \begin{cases} 1 & \text{if } p \in 4\mathbb{N} + 1 \\ i & \text{if } p \in 4\mathbb{N} - 1 \end{cases},$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol (see e.g. [39]).

It is immediate to see that the orthonormal basis $\{\psi_k^a\}_{k \in \mathbb{Z}_p}$ is Fourier connected to the orthonormal basis $\{\varphi'_j\}_{j \in \mathbb{Z}_p}$ given by

$$\varphi'_j = \omega^{aj^2} \varphi_j \quad \forall j,$$

i.e., $\langle \varphi'_j | \psi_k^a \rangle = (1/\sqrt{p}) \omega^{jk}$. Moreover, for $a, b \in \mathbb{Z}_p \setminus \{0\}$, with $a \neq b$, define the rescaled orthonormal bases $\{\psi_j^{a'}\}_{j \in \mathbb{Z}_p}$ and $\{\psi_k^{b'}\}_{k \in \mathbb{Z}_p}$, given by

$$\begin{aligned} \psi_j^{a'} &= \omega^{-4^{-1}j^2(b-a)^{-1}} \psi_j^a \\ \psi_k^{b'} &= \omega^{k^2(b-a)} \left(\frac{b-a}{p}\right) \psi_{2k(b-a)}^b \times \begin{cases} 1 & \text{if } p \in 4\mathbb{N} + 1 \\ -i & \text{if } p \in 4\mathbb{N} - 1 \end{cases} \end{aligned}$$

(Here $\omega^{x^{-1}}$ means ‘ ω to the inverse of x in the field $\mathbb{Z}_p$ ’ and should not be confused with $e^{\frac{2\pi i}{px}}$). Then, an easy computation using the Gauss formula (42) yields

$$\langle \psi_h^{b'} | \psi_k^{a'} \rangle = \frac{1}{\sqrt{p}} \omega^{-hk},$$

which shows that also $\{\psi_j^{a'}\}_{j \in \mathbb{Z}_p}$ and $\{\psi_k^{b'}\}_{k \in \mathbb{Z}_p}$ are Fourier connected.

More generally, one can start from a complementary pair of observables, which means that $\{\varphi_j\}_j$ and $\{\psi_k\}_k$ are mutually unbiased but not necessarily Fourier connected. Obviously, we can still ask similar questions on joint measurements. Especially, it would be interesting to know whether Proposition 6 is still valid under this more general setting. In other words, the question is whether all complementary pairs are essentially similar with respect to joint measurability

Even if we leave this question open in the general case, we can see that our approach generalizes to a larger domain than we have explicitly used it for. Indeed, all our results are still valid (and with only very slight modifications in some of the proofs) if we consider Fourier transform with respect to a generic abelian group G with order d , i.e., $G = \mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_n}$ for $d_1 + \dots + d_n = d$ and $d_i = p_i^{r_i}$, with p_i prime and r_i integer for all $i = 1, \dots, n$. In this case, $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ with $\dim \mathcal{H}_i = d_i$, a basis $\{\varphi_j^i\}_{j \in \mathbb{Z}_{d_i}}$ is chosen in each factor Hilbert space $\mathcal{H}_i$, and the G -Fourier transform of $\mathcal{H}$ is just the tensor product $\mathcal{F} = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$, where each $\mathcal{F}_i$ is the $\mathbb{Z}_{d_i}$ -Fourier transform in $\mathcal{H}_i$ with respect to the basis $\{\varphi_j^i\}_{j \in \mathbb{Z}_{d_i}}$, as defined in (8). The mutually unbiased bases $\{\varphi_j\}_{j \in \mathbb{Z}_d}$ and $\{\psi_k\}_{k \in \mathbb{Z}_d}$ are replaced by the bases $\{\varphi_{j_1, \dots, j_n}\}_{j_1 \in \mathbb{Z}_{d_1}, \dots, j_n \in \mathbb{Z}_{d_n}}$ and $\{\psi_{k_1, \dots, k_n}\}_{k_1 \in \mathbb{Z}_{d_1}, \dots, k_n \in \mathbb{Z}_{d_n}}$ of $\mathcal{H}$, given by

$$\begin{aligned}\varphi_{j_1, \dots, j_n} &= \varphi_{j_1}^1 \otimes \dots \otimes \varphi_{j_n}^n \\ \psi_{k_1, \dots, k_n} &= \mathcal{F}^*(\varphi_{j_1, \dots, j_n}) = \psi_{k_1}^1 \otimes \dots \otimes \psi_{k_n}^n.\end{aligned}$$

Their associated complementary observables $\mathbf{A}$ and $\mathbf{B}$ are now both defined on G , and given by $\mathbf{A}(j_1, \dots, j_n) = \mathbf{A}(j_1) \otimes \dots \otimes \mathbf{A}(j_n)$ and $\mathbf{B}(k_1, \dots, k_n) = \mathbf{B}(k_1) \otimes \dots \otimes \mathbf{B}(k_n)$. They still satisfy the analogues of the covariance and invariance conditions (10) – (11), if the representations U and V are replaced by suitable tensor products.

To demonstrate that we can now handle larger class of complementary observables, let $\mathcal{H} = \mathbb{C}^4$, choose an orthonormal basis $\{\varphi_j\}_{j \in \{0, \dots, 3\}}$ of $\mathbb{C}^4$ and set

$$\begin{aligned}\psi_0 &= \frac{1}{2}(\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3), & \psi_1 &= \frac{1}{2}(\varphi_0 - \varphi_1 + \varphi_2 - \varphi_3), \\ \psi_2 &= \frac{1}{2}(\varphi_0 + \varphi_1 - \varphi_2 - \varphi_3), & \psi_3 &= \frac{1}{2}(\varphi_0 - \varphi_1 - \varphi_2 + \varphi_3).\end{aligned}$$

Then the observables $\mathbf{A}(j) = |\varphi_j\rangle\langle\varphi_j|$ and $\mathbf{B}(k) = |\psi_k\rangle\langle\psi_k|$ are complementary. They are equally defined by any two orthonormal bases $\{\alpha_j \varphi_j\}_{j \in \{0, \dots, 3\}}$ and $\{\beta_k \psi_k\}_{k \in \{0, \dots, 3\}}$, where α_j, β_k are complex numbers with unit modulus. If some pair of these orthonormal bases were connected by the $\mathbb{Z}_4$ -Fourier transform, then the matrix of their scalar products $[\overline{\alpha_j} \beta_k \langle \varphi_j | \psi_k \rangle]$ should be equal to the $\mathbb{Z}_4$ -Fourier matrix

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

or to a matrix obtained from the above by some permutations of its rows and columns. It is straightforward to verify that the deriving set of equations for α_j and β_k has no solution. However, the matrix of

scalar products $[\langle \varphi_j | \psi_k \rangle]$ is just the Fourier matrix of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.,

$$(43) \quad \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

In other words, the two orthonormal bases are connected by the Fourier transform of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

APPENDIX: CRITERION FOR INFORMATIONAL COMPLETENESS

Theorem 1. *Let C_T be a covariant phase space observable. Then C_T is informationally complete if and only if*

$$(44) \quad \text{tr}[TU_x V_y] \neq 0 \quad \forall x, y \in \mathbb{Z}_d.$$

Our proof of the above theorem relies on the following well known reconstruction formula for the Weyl-Heisenberg group, which is just a special case of orthogonality relations for irreducible representations of compact groups.

Proposition 12. *The following reconstruction formula holds for every $A \in \mathcal{L}(\mathcal{H})$:*

$$(45) \quad \frac{1}{d} \sum_{x, y \in \mathbb{Z}_d} \text{tr}[AV_y^* U_x^*] U_x V_y = A.$$

Proof. For all $h, k \in \mathbb{Z}_d$, we have $\langle \varphi_k | U_x V_y \varphi_h \rangle = \omega^{hy} \delta_{h+x, k}$. Thus, for all $h, k, m, n \in \mathbb{Z}_d$, we obtain

$$\begin{aligned} \left\langle \varphi_k \left| \left[\sum_{x, y \in \mathbb{Z}_d} \text{tr}[\varphi_m \langle \varphi_n | V_y^* U_x^*] U_x V_y \right] \varphi_h \right\rangle \right. \\ &= \sum_{x, y \in \mathbb{Z}_d} \langle U_x V_y \varphi_n | \varphi_m \rangle \langle \varphi_k | U_x V_y \varphi_h \rangle \\ &= \sum_{x, y \in \mathbb{Z}_d} \omega^{(h-n)y} \delta_{h+x, k} \delta_{n+x, m} \\ &= \sum_{y \in \mathbb{Z}_d} \omega^{(h-n)y} \delta_{h-n, k-m} \\ &= d \delta_{h, n} \delta_{h-n, k-m} \\ &= d \delta_{h, n} \delta_{k, m} \\ &= d \langle \varphi_k | \varphi_m \rangle \langle \varphi_n | \varphi_h \rangle, \end{aligned}$$

which proves (45) for $A = |\varphi_m\rangle\langle\varphi_n|$. Since every $A \in \mathcal{L}(\mathcal{H})$ is a linear combination of this type of operators, the claim follows. $\square$

Proof of Theorem 1. Let $\ell(\mathbb{Z}_d^2)$ be the linear space of complex functions on $\mathbb{Z}_d^2 \equiv \mathbb{Z}_d \times \mathbb{Z}_d$. We recall that, by Proposition 5.1 in [40], $\mathbf{C}_T$ is informationally complete if and only if the linear map

$$V_T : \mathcal{L}(\mathcal{H}) \rightarrow \ell(\mathbb{Z}_d^2), \quad [V_T(A)](x, y) = \text{tr} [\mathbf{C}_T(x, y)A]$$

is injective. Since the dimensions of $\mathcal{L}(\mathcal{H})$ and $\ell(\mathbb{Z}_d^2)$ are both d^2 , we conclude that $\mathbf{C}_T$ is informationally complete if and only if V_T is an isomorphism.

We define the following three linear maps

$$\begin{aligned} \Phi : \mathcal{L}(\mathcal{H}) &\rightarrow \mathcal{H} \otimes \mathcal{H}, & \Phi(A) &= \frac{1}{d} \sum_{x, y \in \mathbb{Z}_d} \text{tr} [AV_y^* U_x^*] \varphi_x \otimes \varphi_y \\ M_T : \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H}, & M_T(\varphi_x \otimes \varphi_y) &= \text{tr} [TU_x V_y] \varphi_x \otimes \varphi_y \\ R : \mathcal{H} \otimes \mathcal{H} &\rightarrow \ell(\mathbb{Z}_d^2), & R\phi(x, y) &= \langle \varphi_y \otimes \varphi_x | \phi \rangle. \end{aligned}$$

The map R is clearly a linear isomorphism, Φ is a linear isomorphism by Proposition 12, and M_T is a linear isomorphism if and only if (44) holds. We now evaluate the composition map $R(\mathcal{F} \otimes \mathcal{F}^*)M_T\Phi$. For all $A \in \mathcal{L}(\mathcal{H})$, we obtain

$$\begin{aligned} [R(\mathcal{F} \otimes \mathcal{F}^*)M_T\Phi(A)](h, k) &= \\ &= \frac{1}{d} \sum_{x, y \in \mathbb{Z}_d} \text{tr} [TU_x V_y] \text{tr} [AV_y^* U_x^*] \langle \varphi_k \otimes \varphi_h | (\mathcal{F} \otimes \mathcal{F}^*)(\varphi_x \otimes \varphi_y) \rangle \\ &= \frac{1}{d^2} \sum_{x, y \in \mathbb{Z}_d} \text{tr} [TU_x V_y] \text{tr} [AV_y^* U_x^*] \omega^{yh-xk} \\ &= \frac{1}{d^2} \sum_{x', y' \in \mathbb{Z}_d} \omega^{y'h-x'k} \text{tr} [TU_h^* U_{x'} V_k^* V_{y'}] \text{tr} [AV_k V_{y'}^* U_h U_{x'}^*] \\ &= \frac{1}{d^2} \sum_{x', y' \in \mathbb{Z}_d} \text{tr} [TU_h^* V_k^* U_{x'} V_{y'}] \text{tr} [AV_k U_h V_{y'}^* U_{x'}^*] \\ &= \frac{1}{d} \text{tr} [TU_h^* V_k^* AV_k U_h] \\ &= [V_T(A)](h, k) \end{aligned}$$

(in the third equality we set $x = x' - h$, $y = y' - k$, in the fourth we used the commutation relation for U and V , and in the fifth we applied the reconstruction formula (45)). As the map $R(\mathcal{F} \otimes \mathcal{F}^*)M_T\Phi$ is an isomorphism if and only if (44) holds, the same is true for the map V_T , and the theorem follows. $\square$

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REFERENCES

- [1] E. Prugovečki. Information-theoretical aspects of quantum measurements. *Int. J. Theor. Phys.*, 16:321–331, 1977.
- [2] P. Busch and P. Lahti. The determination of the past and the future of a physical system in quantum mechanics. *Found. Phys.*, 19:633–678, 1989.
- [3] P. Busch. Informationally complete sets of physical quantities. *Internat. J. Theoret. Phys.*, 30(9):1217–1227, 1991.
- [4] T. Opatrný, V. Bužek, J. Bajer, and G. Drobný. Propensities in discrete phase spaces: Q function of a state in a finite-dimensional Hilbert space. *Phys. Rev. A*, 52:2419–2428, 1995.
- [5] A. Vourdas. Phase space methods for finite quantum systems. *Rep. Math. Phys.*, 40:367–371, 1997.
- [6] I. D. Ivanović. Geometrical description of quantal state determination. *J. Phys. A: Math. Gen.*, 14:3241–3245, 1981.
- [7] W.K. Wootters. Quantum mechanics without probability amplitudes. *Found. Phys.*, 16:391–405, 1986.
- [8] W.K. Wootters and B.D. Fields. Optimal state-determination by mutually unbiased measurements. *Ann. Physics*, 191:363–381, 1989.
- [9] S. Brierley and S. Weigert. Maximal sets of mutually unbiased quantum states in dimension 6. *Phys. Rev. A*, 78:042312, 2008.
- [10] P. Raynal, X. Lü, and B.-G. Englert. Mutually unbiased bases in six dimensions: The four most distant bases. *Phys. Rev. A*, 83:062303, 2011.
- [11] P. Busch. Unsharp reality and joint measurements for spin observables. *Phys. Rev. D*, 33:2253–2261, 1986.
- [12] C. Carmeli, T. Heinosaari, and A. Toigo. Sequential measurements of conjugate observables. *J. Phys. A: Math. Theor.*, 44:285304, 2011.
- [13] J.M. Renes, R. Blume-Kohout, A.J. Scott, and C.M. Caves. Symmetric informationally complete quantum measurements. *J. Math. Phys.*, 45:2171–2180, 2004.
- [14] D.M. Appleby. Symmetric informationally complete-positive operator valued measures and the extended clifford group. *J. Math. Phys.*, 46:052107, 2005.
- [15] A.J. Scott and M. Grassl. Symmetric informationally complete positive-operator-valued measures: A new computer study. *J. Math. Phys.*, 51:042203, 2010.
- [16] M. Shalaby and A. Vourdas. Tomographically complete sets of orthonormal bases in finite systems. *J. Phys. A: Math. Theor.*, 44:345303, 2011.
- [17] P. Busch, M. Grabowski, and P.J. Lahti. *Operational Quantum Physics*. Springer-Verlag, Berlin, 1997. second corrected printing.
- [18] W.M. de Muynck. *Foundations of Quantum Mechanics, an Empiricist Approach*. Kluwer Academic Publishers, Dordrecht, 2002.

- [19] M. Singer and W. Stulpe. Phase-space representations of general statistical physical theories. *J. Math. Phys.*, 33:131–142, 1992.
- [20] P. Lahti and S. Pulmannová. Coexistent observables and effects in quantum mechanics. *Rep. Math. Phys.*, 39:339–351, 1997.
- [21] S.T. Ali, C. Carmeli, T. Heinosaari, and A. Toigo. Commutative povms and fuzzy observables. *Found. Phys.*, 39:593–612, 2009.
- [22] E.B. Davies. *Quantum Theory of Open Systems*. Academic Press, London, 1976.
- [23] M. Ozawa. Quantum measuring processes of continuous observables. *J. Math. Phys.*, 25:79–87, 1984.
- [24] M. Hayashi. *Quantum information*. Springer-Verlag, Berlin, 2006. Translated from the 2003 Japanese original.
- [25] E.B. Davies and J.T. Lewis. An operational approach to quantum probability. *Comm. Math. Phys.*, 17:239–260, 1970.
- [26] P. Busch, G. Cassinelli, and P. Lahti. On the quantum theory of sequential measurements. *Found. Phys.*, 20:757–775, 1990.
- [27] P. Busch and T. Heinosaari. Approximate joint measurements of qubit observables. *Quant. Inf. Comp.*, 8:0797–0818, 2008.
- [28] J. Schwinger. Unitary operator bases. *Proc. Nat. Acad. Sci. U.S.A.*, 46:570–579, 1960.
- [29] K. Kraus. Complementary observables and uncertainty relations. *Phys. Rev. D*, 35:3070–3075, 1987.
- [30] C. Carmeli, T. Heinonen, and A. Toigo. Position and momentum observables on $\mathbb{R}$ and on $\mathbb{R}^3$. *J. Math. Phys.*, 45:2526–2539, 2004.
- [31] P. Stano, D. Reitzner, and T. Heinosaari. Coexistence of qubit effects. *Phys. Rev. A*, 78:012315, 2008.
- [32] P. Busch and H.-J. Schmidt. Coexistence of qubit effects. *Quantum Inf. Process.*, 9:143–169, 2010.
- [33] S. Yu, N.-L. Liu, L. Li, and C.H. Oh. Joint measurement of two unsharp observables of a qubit. *Phys. Rev. A*, 81:062116, 2010.
- [34] T. Miyadera and H. Imai. Heisenberg’s uncertainty principle for simultaneous measurement of positive-operator-valued measures. *Phys. Rev. A*, 78:052119, 2008.
- [35] S.T. Ali and E. Prugovečki. Systems of imprimitivity and representations of quantum mechanics on fuzzy phase spaces. *J. Math. Phys.*, 18:219–228, 1977.
- [36] G. Cassinelli, E. De Vito, P. Lahti, and A. Levrero. Phase space observables and isotypic spaces. *J. Math. Phys.*, 41:5883–5896, 2000.
- [37] G.M. D’Ariano, P. Perinotti, and M.F. Sacchi. Informationally complete measurements and group representation. *J. Opt. B: Quantum Semiclass. Opt.*, 6:S487–S491, 2004.
- [38] T. Heinosaari and M.M. Wolf. Nondisturbing quantum measurements. *J. Math. Phys.*, 51:092201, 2010.
- [39] B.C. Berndt and R.J. Evans. The determination of Gauss sums. *Bull. Amer. Math. Soc.*, 2:107–129, 1981.
- [40] P. Busch, G. Cassinelli, and P.J. Lahti. Probability structures for quantum state spaces. *Rev. Math. Phys.*, 7:1105–1121, 1995.