Covariant mutually unbiased bases

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Quadratures in quantum homodyne tomography

Q, P : position and momentum operators on $\mathcal{H} = L^2(\mathbb{R})$

$W(q,p) = \mathrm{e}^{i(pQ-qP)}$: Weyl operators

They satisfy the Weyl commutation relations

$$W(\mathbf{u})W(\mathbf{u}') = \mathrm{e}^{-i\frac{S(\mathbf{u},\mathbf{u}')}{2}} W(\mathbf{u}+\mathbf{u}') = \mathrm{e}^{-iS(\mathbf{u},\mathbf{u}')} W(\mathbf{u}')W(\mathbf{u})$$

with the symplectic form

$$S(\mathbf{u},\mathbf{u}') = u_1 u_2' - u_1' u_2$$

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$\mathcal{D} = \{D \subset \mathbb{R}^2 \mid D \text{ is a 1-dimensional subspace}\}$

$\Rightarrow$ $W|_D$ is a unitary representation of $(D, +)$

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$\mathcal{D} = \{D \subset \mathbb{R}^2 \mid D = \mathbb{R}\mathbf{u}_\theta,\ \mathbf{u}_\theta = (\sin\theta, -\cos\theta)\}$

$\Rightarrow\ W(r\mathbf{u}_\theta) = \int \mathrm{e}^{-irx} \mathsf{Q}_\theta(\mathrm{d}x)\ \ \forall r \in \mathbb{R}$ with Q_θ PVM on $\mathbb{R}$

Q_θ : quadrature observable along the direction θ

Quadratures in quantum homodyne tomography

Covariance properties wrt translations:

$$W(\mathbf{u}) \mathsf{Q}_\theta(X) W(\mathbf{u})^* = \mathsf{Q}_\theta(X + u_1 \cos\theta + u_2 \sin\theta)$$

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Covariance properties wrt symplectic transformations:

$$U(A)\mathrm{Q}_\theta(X)U(A)^* = \mathrm{Q}_{\theta_A}(c_A X)$$

where $A \in \mathrm{SL}(2,\mathbb{R})$ and U is the *metaplectic representation*

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Restricting to $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$

$$\mathrm{e}^{i\phi N}\mathrm{Q}_\theta(X)\mathrm{e}^{-i\phi N} = \mathrm{Q}_{\theta+\phi}(X)$$

where

$$N = \frac{1}{2}(Q^2 + P^2 - 1)$$

Quadratures in finite dimensions

$\mathbb{F} = \mathbb{Z}_p$ with p odd prime

$$[W(\mathbf{u})f](x) = \mathrm{e}^{\frac{2\pi i}{p}(2^{-1}u_1 u_2 - u_1 x)} f(x - u_2) \qquad \forall f \in \mathcal{H} \equiv \ell^2(\mathbb{F})$$

The Weyl commutation relations are

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$\mathcal{D} = \{D \subset \mathbb{F}^2 \mid D = \mathbb{F}\mathbf{u}_D,\ \mathbf{u}_D \neq \mathbf{0}\}$

$$\Rightarrow \ W(r\mathbf{u}_D) = \textstyle\sum_{x\in\mathbb{F}} \mathrm{e}^{-\frac{2\pi i}{p}rx} \mathrm{Q}_D(x) \ \ \forall r \in \mathbb{F} \quad \text{with } \mathrm{Q}_D \text{ PVM on } \mathbb{F}$$

Q_D : quadrature observable along the direction D

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Quadratures in finite dimensions

Properties of quadratures in finite dimensions:

$$\mathrm{tr}\left[\mathsf{Q}_D(x)\mathsf{Q}_{D'}(y)\right] = \frac{1}{p} \quad \text{if} \quad D \neq D'$$

$$\Rightarrow \quad \mathsf{Q}_D(x) = |\phi_D(x)\rangle\langle\phi_D(x)| \quad \text{with} \quad \langle\, \phi_D(x)\,|\,\phi_{D'}(y)\,\rangle = \frac{1}{p}$$

$|\mathcal{D}| = p + 1 \;\Rightarrow\;$ the PVMs $\{\mathsf{Q}_D \mid D \in \mathcal{D}\}$ give a maximal set of MUBs

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Everything extends to any finite field with odd characteristic

But what happens if $p = 2$?

What are the symmetries of finite quadrature observables?

And what is the finite analogue of $\mathrm{SO}(2)$?

Finite phase-space

$\mathbb{F}$: finite field with characteristic p

V : 2-dimensional vector space over $\mathbb{F}$

Ω : 2-dimensional affine space with translation group V (phase-space)

$x, y \in \Omega \;\Rightarrow\; \exists! \mathbf{u}_{x,y}$ s.t. $y = x + \mathbf{u}_{x,y}$

$L(\Omega)$: affine lines of Ω

$\mathcal{D} = \{D \subset V \mid \dim_{\mathbb{F}} D = 1\}$: directions of Ω (they are $|\mathbb{F}| + 1$)

$L_D(\Omega)$: lines with direction $D \in \mathcal{D}$

$\mathfrak{l} \in L_D(\Omega) \;\Rightarrow\; \mathfrak{l} = x + D = \{x + \mathbf{d} \mid \mathbf{d} \in D\}$ with $x \in \Omega$

Quadrature systems

Definition

A *quadrature system* (QS) for the affine space (Ω, V) acting on the Hilbert space $\mathcal{H}$ is a map $\mathrm{Q} : L(\Omega) \to \mathcal{L}(\mathcal{H})$ such that

(i) $\mathrm{Q}(\mathfrak{l})$ is a rank-1 orthogonal projection for all $\mathfrak{l} \in L(\Omega)$

(ii) for all $D \in \mathcal{D}$
$$\sum_{\mathfrak{l} \in L_D(\Omega)} \mathrm{Q}(\mathfrak{l}) = \mathbb{1}$$

(iii) for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \neq D_2$,
$$\mathrm{tr}\,[\mathrm{Q}(\mathfrak{l}_1)\mathrm{Q}(\mathfrak{l}_2)] = \frac{1}{|\mathbb{F}|} \qquad \text{if } \mathfrak{l}_1 \in L_{D_1}(\Omega) \text{ and } \mathfrak{l}_2 \in L_{D_2}(\Omega)$$

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(i), (ii) $\Rightarrow$ $\dim \mathcal{H} = |\mathbb{F}|$

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(i), (ii) $\Rightarrow$ $\mathsf{Q}_D := \mathsf{Q}|_{L_D(\Omega)}$ is a PVM on $L_D(\Omega)$

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(iii) $\Rightarrow$ the PVMs $\{\mathsf{Q}_D \mid D \in \mathcal{D}\}$ project on a maximal set of MUBs

Equivalence of quadrature systems

Definition

Two QSs Q_1 and Q_2 acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are *equivalent* if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\mathsf{Q}_2(\mathfrak{l}) = U \mathsf{Q}_1(\mathfrak{l}) U^* \qquad \forall \mathfrak{l} \in L(\Omega)$$

Our task is to classify equivalence classes of quadrature systems having different symmetry properties

Symmetries of the phase-space

$\mathrm{GL}(V) \rtimes V$: affine group

Fixed an *origin* $o \in \Omega$, an element $(A, \mathbf{v}) \in \mathrm{GL}(V) \rtimes V$ acts on Ω:

$$(A, \mathbf{v}) \cdot x = o + A(\mathbf{u}_{o,x} + \mathbf{v}) \qquad \forall x \in \Omega$$

and on $L(\Omega)$:

$$(A, \mathbf{v}) \cdot (x + D) = (A, \mathbf{v}) \cdot x + AD \qquad \forall x + D \in L(\Omega)$$

The action of V on $L(\Omega)$ preserves the sets $L_D(\Omega)$

Covariant quadrature systems

If Q is a QS acting on $\mathcal{H}$ and $(A, \mathbf{v}) \in \mathrm{GL}(V) \rtimes V$, then

$$\mathrm{Q}_{(A,\mathbf{v})} : L(\Omega) \to \mathcal{L}(\mathcal{H}) \qquad \mathrm{Q}_{(A,\mathbf{v})}(\mathfrak{l}) = \mathrm{Q}((A, \mathbf{v}) \cdot \mathfrak{l})$$

is another QS acting on $\mathcal{H}$

Definition

Let $G \subseteq \mathrm{GL}(V) \rtimes V$ be any subgroup. A QS Q is *G-covariant* if

$$\mathrm{Q} \sim \mathrm{Q}_{(A,\mathbf{v})} \qquad \forall (A, \mathbf{v}) \in G$$

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$$\mathrm{Q}_{(A,\mathbf{v})}(\mathfrak{l}) = U(A, \mathbf{v})\mathrm{Q}(\mathfrak{l})U(A, \mathbf{v})^* \qquad \forall \mathfrak{l} \in L(\Omega),\ (A, \mathbf{v}) \in G$$

where U is a unitary projective representation of G *associated* with Q

V-covariant quadrature systems

Theorem

Suppose Q *is a V-covariant QS, and let $o \in \Omega$. Then $\exists!$ projective representation W of V associated with* Q *and such that*

(a) *for all $D \in \mathcal{D}$, the restriction $W|_D$ is an ordinary representation of $(D, +)$*

(b) *for all $D \in \mathcal{D}$*

$$W(\mathbf{d})\mathrm{Q}(o + D) = \mathrm{Q}(o + D) \qquad \forall \mathbf{d} \in D$$

Moreover, W satisfy the commutation relation

$$W(\mathbf{u})W(\mathbf{u}') = \mathrm{e}^{-\frac{2\pi i}{p} \operatorname{Tr} S(\mathbf{u},\mathbf{u}')} W(\mathbf{u}')W(\mathbf{u}) \qquad \forall \mathbf{u}, \mathbf{u}' \in V$$

where S is a symplectic form on V uniquely determined by Q

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W : *Weyl system* (WS) associated with Q and centered at o

$$W(\mathbf{u})\mathrm{Q}(o + D) = \mathrm{Q}(o + D) \qquad \forall \mathbf{u} \in D$$

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(b) *for all $D \in \mathcal{I}$* …

S : *symplectic form* induced by Q

(i) S is nonzero

(ii) $S(\mathbf{u}, \mathbf{u}) = 0$ for all $\mathbf{u} \in V$

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Inequivalent V-covariant QSs may induce the same symplectic form...

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... but it turns out that their associated *centered* WSs are inequivalent

V-covariant quadrature systems and Weyl multipliers

Definition

A multiplier m on $(V, +)$ is a *Weyl multiplier* for S if

(i) for all $D \in \mathcal{D}$, the restriction $m|_{D \times D} = 1$

(ii) $\overline{m(\mathbf{u}, \mathbf{u}')} m(\mathbf{u}', \mathbf{u}) = \mathrm{e}^{-\frac{2\pi i}{p} \operatorname{Tr} S(\mathbf{u}, \mathbf{u}')}$ for all $\mathbf{u}, \mathbf{u}' \in V$

The multiplier of a WS is a Weyl multiplier

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Theorem

The correspondence between V-covariant QSs and Weyl multipliers is one-to-one and onto

Equivalence classes of V-covariant quadratures

For each symplectic form S, there are $|\mathbb{F}|^{|\mathbb{F}|-1}$ different Weyl multipliers
$\Rightarrow$ there are $|\mathbb{F}|^{|\mathbb{F}|-1}$ inequivalent V-covariant QSs inducing S

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$\Rightarrow$ there are $(|\mathbb{F}| - 1)|\mathbb{F}|^{|\mathbb{F}|-1}$ inequivalent V-covariant QSs in total

But...

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Theorem

Let Q_1 and Q_2 be any two V-covariant QSs, with Q_i acting on $\mathcal{H}_i$. Then there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\operatorname{ran} \mathsf{Q}_1 = U(\operatorname{ran} \mathsf{Q}_2)U^*$$

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$$\{\mathsf{Q}_2(\mathfrak{l}) \mid \mathfrak{l} \in L(\Omega)\} = U\{\mathsf{Q}_1(\mathfrak{l}) \mid \mathfrak{l} \in L(\Omega)\}U^*)U^*$$

Covariance with respect to larger groups

Theorem

Suppose $G_0 \subset \mathrm{GL}(V)$. There exist $(G_0 \rtimes V)$-covariant QSs only if

$$G_0 \subseteq \mathrm{SL}(V) = \{A \in \mathrm{GL}(V) \mid \det A = 1\}$$

But it may happen that there do not exist $(G_0 \rtimes V)$-covariant QSs!

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But it may happen that there do not exist $(G_0 \rtimes V)$-covariant QSs!

Theorem

There exist $(\mathrm{SL}(V) \rtimes V)$*-covariant QSs if and only if* $p \neq 2$.
In this case, for any symplectic form S *on* V *there is exactely one equivalence class of* $(\mathrm{SL}(V) \rtimes V)$*-covariant QSs that induce* S

Nonsplit toruses

Definition

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The action of a maximal nonsplit torus on the set $\mathcal{D}$ of directions

- is free and transitive if $p = 2$;
- has two orbits with $(|\mathbb{F}| + 1)/2$ elements in each orbit if $p \neq 2$.

Covariance with respect to nonsplit toruses

Theorem

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Let T be a maximal nonsplit torus, and suppose Q *is a $T \rtimes V$-covariant QS. Let W be the Weyl system associated with* Q *and centered at the origin $o \in \Omega$. Let m be its Weyl multiplier. Then, the projective representation U of $T \rtimes V$ associated with* Q *is given by*

$$U(A, \mathbf{v}) = \frac{c(A)}{|\mathbb{F}|} \sum_{\mathbf{u} \in V} m(\mathbf{u}, (A - I)^{-1}\mathbf{u}) W(\mathbf{u}) W(\mathbf{v}) \qquad \forall A \in T \setminus \{I\},$$

where c(A) is an arbitrary phase factor.

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   $\Rightarrow$ 2 inequivalent V-covariant QSs Q and $\overline{\text{Q}}$

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3. there are 2 Weyl multipliers on $(V, +)$: m and $\overline{m}$
   $\Rightarrow$ 2 inequivalent V-covariant QSs Q and $\overline{\mathrm{Q}}$
4. $\mathrm{SL}(V) = H \rtimes T$, with T maximal nonsplit torus
5. Q and $\overline{\mathrm{Q}}$ are $(T \rtimes V)$-covariant

References

C. Carmeli, J. Schultz and A. Toigo, *Covariant mutually unbiased bases*, arXiv:1504.06415v2

Thank you!

Projective representations

A *projective representation* U of G satisfies

$$U(g_1 g_2) = m(g_1, g_2) \qquad \forall g_1, g_2 \in G$$

where

$$m : G \times G \to \{z \in \mathbb{C} \mid |z| = 1\}$$

with

$$m(g_1, g_2 g_3) m(g_2, g_3) = m(g_1 g_2, g_3) m(g_1, g_2) \qquad \forall g \in G$$

is the *multiplier* of U