COMPLETELY POSITIVE TRANSFORMATIONS OF QUANTUM OPERATIONS

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Quantum supermaps are higher-order maps transforming quantum operations into quantum operations and satisfying suitable requirements of normality and complete positivity. Here we present the extension of the theory of quantum supermaps, originally formulated in the finite dimensional setting, to the case of higher-order maps transforming quantum operations on generic von Neumann algebras. In this setting, we provide two dilation theorems for quantum supermaps that are the analogues of the Stinespring and Radon-Nikodym theorems for quantum operations. A structure theorem for probability measures with values in the set of quantum supermaps is also illustrated. Finally, some applications are given, and in particular it is shown that all the supermaps defined in this paper can be implemented by connecting quantum devices in quantum circuits.

Keywords: Completely positive and completely bounded maps; dilation theorems; Stinespring representation; quantum information theory.

1. Introduction

Quantum supermaps[1,2] are the most general admissible transformations of quantum devices. Mathematically, the action of a quantum device is associated to a set of completely positive trace non-increasing maps, called *quantum operations*,[3,4] which transform the states of an input quantum system into states of an output quantum system. A quantum supermap is then a higher-order linear map that transforms quantum operations into quantum operations. The theory of quantum supermaps, developed in Refs. 1,2 for finite dimensional quantum systems, has proven to be a powerful tool for the treatment of many advanced topics in quantum information theory,[5–10] including in particular the optimal cloning and the optimal learning of unitary transformations[11,12] and quantum measurements.[13,14] Moreoever, quantum supermaps are interesting for the foundations of Quantum Mechanics as they are the possible dynamics in a toy model of non-causal theory,[15] which has a quartic relation between the number of distinguishable states and the number of parameters needed to specify a state.[16] Quantum supermaps also attracted interest in the mathematical physics literature, as they suggested the study of a general class of positive maps between convex subsets of the state space.[17]

Originally, the definition and the main theorems on quantum supermaps were presented in the context of full matrix algebras describing finite dimensional quantum systems.[1,2] In this paper we will present their extension to the case where the input of the quantum operations is allowed to be a generic von Neumann algebra and the output is the C^*-algebra of the bounded operators on an arbitrary Hilbert space. This generalization is useful for applications, because on the one hand it allows to treat quantum systems with infinite dimension, on the other hand it includes transformations of quantum measuring devices, i.e. maps from the commutative algebra of functions on the outcome space to the full operator algebra on the Hilbert space of the measured system (in the Heisenberg picture).

Quantum supermaps on finite dimensional quantum systems are defined axiomatically as completely positive linear maps transforming quantum operations into quantum operations (see Refs. 1,2 for the physical motivation of linearity and complete positivity). A quantum supermap is *deterministic* if it transforms quantum channels (i.e. unital completely positive maps[18]) into quantum channels. The following dilation theorem can be proved for deterministic supermaps in finite dimension:[1,2] denoting by $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$ the C^*-algebras of the linear operators on the finite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, and writing $\mathrm{CP}\left(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K})\right)$ for the set of completely positive maps sending $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\mathcal{K})$, we have that any deterministic supermap S transforming quantum operations in

$\mathrm{CP}\,(\mathcal{L}(\mathcal{H}_1),\mathcal{L}(\mathcal{K}_1))$ to quantum operations in $\mathrm{CP}\,(\mathcal{L}(\mathcal{H}_2),\mathcal{L}(\mathcal{K}_2))$ has the following form:

$$[\mathsf{S}(\mathcal{E})](A) = V_1^*\left[(\mathcal{E}\otimes\mathcal{I}_{\mathcal{V}_1})(V_2^*(A\otimes I_{\mathcal{V}_2})V_2)\right]V_1 \quad \forall A\in\mathcal{L}(\mathcal{H}_2) \tag{1}$$

for all $\mathcal{E}\in\mathrm{CP}\,(\mathcal{L}(\mathcal{H}_1),\mathcal{L}(\mathcal{K}_1))$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are two ancillary separable Hilbert spaces, $V_1:\mathcal{K}_2\to\mathcal{K}_1\otimes\mathcal{V}_1$ and $V_2:\mathcal{H}_1\otimes\mathcal{V}_1\to\mathcal{H}_2\otimes\mathcal{V}_2$ are isometries, $\mathcal{I}_{\mathcal{V}_1}$ is the identity channel on $\mathcal{L}(\mathcal{V}_1)$ and $I_{\mathcal{V}_2}$ is the identity operator on $\mathcal{V}_2$. In the Schrödinger (or predual) picture, eq. (1) can be rewritten

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \mathrm{tr}_{\mathcal{V}_2}\left[V_2(\mathcal{E}_*\otimes\mathcal{I}_{\mathcal{V}_1})(V_1\rho V_1^*)V_2^*\right] \quad \text{for all states } \rho \text{ on } \mathcal{K}_2\,,$$

and thus shows that the most general way to transform a quantum operation consists in

(1) applying an invertible transformation (corresponding to the isometry V_1), which transforms the system $\mathcal{K}_2$ into the composite system $\mathcal{K}_1\otimes\mathcal{V}_1$;
(2) using the input device $\mathcal{E}_*$ on system $\mathcal{K}_1$, thus transforming it into system $\mathcal{H}_1$, while doing nothing on $\mathcal{V}_1$;
(3) applying an invertible transformation (corresponding to the isometry V_2), which transforms the composite system $\mathcal{H}_1\otimes\mathcal{V}_1$ into the composite system $\mathcal{H}_2\otimes\mathcal{V}_2$;
(4) discarding system $\mathcal{V}_2$ (mathematically, taking the partial trace over $\mathcal{V}_2$).

In this paper we will first present an extension of eq. (1) to the case of quantum supermaps acting on quantum operations with generic von Neumann algebras in the input, in particular removing all requirements of finite dimensionality. It will turn out that the definition of a suitable notion of *normality* of supermaps is the key point in order to extend the main theorems in the infinite dimensional case.[19,20] Then, as a second step we will state a Radon-Nikodym theorem for probabilistic supermaps, namely supermaps that are dominated by deterministic supermaps. The class of probabilistic supermaps is particularly interesting for physical applications, as such maps naturally appear in the description of quantum circuits that are designed to test properties of physical devices.[1,2,21] Higher-order quantum measurements are indeed described by *quantum superinstruments*, which are the generalization of the quantum instruments of Davies and Lewis.[22] The third main result exposed in the paper will then be a dilation theorem for quantum superinstruments, in analogy with Ozawa's dilation theorem for ordinary instruments.[23]

The present paper is intended as a review of Ref. 19 and our recent still unpublished work Ref. 20. As such, it contains a survey of the main results and applications, but defers the interested reader to Refs. 19,20 for a more detailed and technical exposition and for the complete proofs of the statements. The material of the paper is organized as follows. In Section 2 we fix the elementary definitions and notations, and state or recall some basic facts needed in the rest of the paper. In particular, in Section 2.1 we extend the notion of increasing sequences from positive operators to normal completely positive maps, while Section 2.2 contains some elementary results about the tensor product of weak*-continuous maps. In Section 3 we define normal completely positive supermaps and provide some examples. In Section 4 we state and comment the two main results of the paper, i.e our dilation Theorem 4.1 for deterministic supermaps and its extension to probabilistic supermaps contained in Theorem 4.2 (Radon-Nikodym theorem for probabilistic supermaps). As an application of Theorem 4.1, in Section 5 we show that every deterministic supermap transforming measurements into quantum operations can be realized by connecting devices in a quantum circuit. We then define quantum superinstruments in Section 6 and state a dilation theorem for them which is the generalization of classical Ozawa's result for ordinary instruments (see in particular Proposition 4.2 in Ref. 23). Finally, in Section 7 we apply the dilation theorem for quantum superinstruments in order to show how every abstract superinstrument describing a measurement on a quantum measuring device can be realized in a circuit.

2. Notations and preliminary results

In this paper, we will always mean by *Hilbert space* a complex and separable Hilbert space, with scalar product $\langle \cdot , \cdot \rangle$ linear in the second entry. If $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces, we denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ endowed with the uniform norm $\|\cdot\|_\infty$. If $\mathcal{H} = \mathcal{K}$, the shortened notation $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ will be used, and $I_\mathcal{H}$ will be the identity operator in $\mathcal{L}(\mathcal{H})$. The linear space $\mathcal{L}(\mathcal{H})$ is ordered in the usual way. We denote by $\leq$ the order relation in $\mathcal{L}(\mathcal{H})$, and by $\mathcal{L}(\mathcal{H})_+$ the cone of positive operators.

Following Ref. 24 (see Definition 3.2 p. 72), by *von Neumann algebra* we mean a $*$-subalgebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ such that $\mathcal{M} = (\mathcal{M}')'$, where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ in $\mathcal{L}(\mathcal{H})$. Note that, since all Hilbert spaces considered in the paper are separable, the von Neumann algebras considered here are those that are sometimes called *separable* in the literature. When $\mathcal{M}$ is regarded as an abstract von Neumann algebra (i.e. without reference to the representing Hilbert space $\mathcal{H}$), we will write its identity element $I_\mathcal{M}$ instead of $I_\mathcal{H}$. As usual, we define $\mathcal{M}_+ := \mathcal{M} \cap \mathcal{L}(\mathcal{H})_+$. The identity map on $\mathcal{M}$

will be denoted by $\mathcal{I}_{\mathcal{M}}$, and, when $\mathcal{M} \equiv \mathcal{L}(\mathcal{H})$, the abbreviated notation $\mathcal{I}_{\mathcal{H}} := \mathcal{I}_{\mathcal{L}(\mathcal{H})}$ will be used.

The algebraic tensor product of linear spaces U, V will be written $U \hat{\otimes} V$, while the notation $\mathcal{H} \otimes \mathcal{K}$ will be reserved to denote the Hilbert space tensor product of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The inclusion $\mathcal{H} \hat{\otimes} \mathcal{K} \subset \mathcal{H} \otimes \mathcal{K}$ holds, and it is actually an equality iff $\mathcal{H}$ or $\mathcal{K}$ is finite dimensional.

If $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$, their tensor product $A \otimes B$, which is well defined as a linear map on $\mathcal{H} \hat{\otimes} \mathcal{K}$, uniquely extends to a bounded operator $A \otimes B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ in the usual way (see e.g. p. 183 in Ref. 24). Thus, the algebraic tensor product $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{K})$ can be regarded as a linear subspace of $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$. Even in this case, the equality $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{L}(\mathcal{K}) = \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ holds iff $\mathcal{H}$ or $\mathcal{K}$ is finite dimensional. More generally, let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ and $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$ be two von Neumann algebras. Then, $\mathcal{M} \hat{\otimes} \mathcal{N}$ is a linear subspace of $\mathcal{L}(\mathcal{H} \otimes \mathcal{K})$. Its weak*-closure is the von Neumann algebra $\mathcal{M} \bar{\otimes} \mathcal{N} \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ (see Definition 1.3 p. 183 in Ref. 24). Clearly, $\mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{M} \bar{\otimes} \mathcal{N}$ iff $\mathcal{M}$ or $\mathcal{N}$ is finite dimensional. It is a standard fact that $\mathcal{L}(\mathcal{H}) \bar{\otimes} \mathcal{L}(\mathcal{K}) = \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ (see eq. 10, p. 185 in Ref. 24).

We denote by $M_n(\mathbb{C})$ the linear space of square $n \times n$ complex matrices, which we identify as usual with the space $\mathcal{L}(\mathbb{C}^n)$. If $\mathcal{M}$ is a von Neumann algebra, we write $\mathcal{M}^{(n)} := M_n(\mathbb{C}) \bar{\otimes} \mathcal{M}$, which is a von Neumann algebra contained in $\mathcal{L}(\mathbb{C}^n \otimes \mathcal{H})$. As remarked above, $\mathcal{M}^{(n)}$ coincides with the algebraic tensor product $M_n(\mathbb{C}) \hat{\otimes} \mathcal{M}$. If $\mathcal{E} : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ and $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ are linear operators, we then see that their algebraic tensor product can be regarded as a linear map $\mathcal{E} \otimes \mathcal{F} : \mathcal{M}^{(m)} \to \mathcal{N}^{(n)}$. Since both $\mathcal{M}^{(m)}$ and $\mathcal{N}^{(n)}$ are von Neumann algebras, it makes sense to speak about positivity and boundedness of $\mathcal{E} \otimes \mathcal{F}$. This fact is at the heart of the following two very well known definitions. In them, we use $\mathcal{I}_n$ to denote the identity map on $M_n(\mathbb{C})$, i.e. $\mathcal{I}_n := \mathcal{I}_{M_n(\mathbb{C})}$.

Definition 2.1. Let $\mathcal{M}$, $\mathcal{N}$ be two von Neumann algebras. Then a linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is

- *completely positive (CP)* if the linear map $\mathcal{I}_n \otimes \mathcal{E}$ is positive, i.e. maps ${\mathcal{M}^{(n)}}_+$ into ${\mathcal{N}^{(n)}}_+$, for all $n \in \mathbb{N}$;
- *completely bounded (CB)* if there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\|(\mathcal{I}_n \otimes \mathcal{E})(\tilde{A})\|_\infty \leq C \|\tilde{A}\|_\infty \quad \forall \tilde{A} \in \mathcal{M}^{(n)},$$

  i.e. if the linear map $\mathcal{I}_n \otimes \mathcal{E}$ is bounded from the Banach space $\mathcal{M}^{(n)}$ into the Banach space $\mathcal{N}^{(n)}$ for all $n \in \mathbb{N}$, and the uniform norms of all the maps $\{\mathcal{I}_n \otimes \mathcal{E}\}_{n \in \mathbb{N}}$ are majorized by a constant independent of n.

We recall that a positive linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is *normal* if it preserves the limits of increasing and bounded sequences, i.e. $\mathcal{E}(A_n) \uparrow \mathcal{E}(A)$ in $\mathcal{N}$ for all increasing sequences $\{A_n\}_{n\in\mathbb{N}}$ and A in $\mathcal{M}_+$ such that $A_n \uparrow A$ (as usual, the notation $A_n \uparrow A$ means that A is the *lower upper bound* of the sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$, see e.g. Lemma 1.7.4 in Ref. 25). It is a standard fact that a positive linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is normal if and only if it is weak*-continuous (Theorem 1.13.2 in Ref. 25).

We introduce the following notations

- $\mathrm{CB}\,(\mathcal{M},\mathcal{N})$ is the linear space of completely bounded *and weak*-continuous* maps from $\mathcal{M}$ to $\mathcal{N}$;
- $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ is the set of *normal* completely positive maps from $\mathcal{M}$ to $\mathcal{N}$;
- $\mathrm{CP}_1\,(\mathcal{M},\mathcal{N})$ is the set of *quantum channels* from $\mathcal{M}$ to $\mathcal{N}$, i.e. the subset of elements $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M},\mathcal{N})$ such that $\mathcal{E}(I_\mathcal{M}) = I_\mathcal{N}$.

Remark 2.1. Suppose $\mathcal{M} \subset M_n(\mathbb{C})$ and $\mathcal{N} \subset M_n(\mathbb{C})$. Then the set $\mathrm{CB}\,(\mathcal{M},\mathcal{N})$ coincides with the space of all linear maps from $\mathcal{M}$ to $\mathcal{N}$ (see e.g. Exercise 3.11 in Ref. 26).

It is well known that each CP map is CB (Proposition 3.6 in Ref. 26) and that each CB map is in the the linear span of four CP maps (Theorem 8.5 in Ref. 26). This fact still holds true when one restricts to weak*-continuous maps, as the next theorem shows (see also Ref. 27 and Theorem 2 in Ref. 19 for the particular case $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and $\mathcal{N} = \mathcal{L}(\mathcal{K})$).

Theorem 2.1. *The inclusion* $\mathrm{CP}\,(\mathcal{M},\mathcal{N}) \subset \mathrm{CB}\,(\mathcal{M},\mathcal{N})$ *holds, and* $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ *is a cone in the linear space* $\mathrm{CB}\,(\mathcal{M},\mathcal{N})$*. For* $\mathcal{N} \equiv \mathcal{L}(\mathcal{K})$*, the linear space spanned by* $\mathrm{CP}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$ *coincides with* $\mathrm{CB}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$*. More precisely, if* $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$*, then there exists four maps* $\mathcal{E}_k \in \mathrm{CP}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$ $(k = 0,1,2,3)$ *such that* $\mathcal{E} = \sum_{k=0}^{3} i^k \mathcal{E}_k$*.*

The cone $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ induces a linear ordering in the space $\mathrm{CB}\,(\mathcal{M},\mathcal{N})$, that we will denote by $\preceq$. Namely, given two maps $\mathcal{E},\mathcal{F} \in \mathrm{CB}\,(\mathcal{M},\mathcal{N})$, we will write $\mathcal{E} \preceq \mathcal{F}$ whenever $\mathcal{F} - \mathcal{E} \in \mathrm{CP}\,(\mathcal{M},\mathcal{N})$.

Two of the main features of CB weak*-continuous maps which we will need in the rest of the paper are the following:

- a notion of limit can be defined for a particular class of sequences in $\mathrm{CB}\,(\mathcal{M},\mathcal{N})$, which is the analogue of the lower upper bound for increasing bounded sequences of operators;

- if $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$ are von Neumann algebras, the maps in $\mathrm{CB}\,(\mathcal{M}_1,\mathcal{N}_1)$ and $\mathrm{CB}\,(\mathcal{M}_2,\mathcal{N}_2)$ can be tensored in order to obtain elements of $\mathrm{CB}\,(\mathcal{M}_1\bar{\otimes}\mathcal{M}_2,\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)$.

As these concepts are the main two ingredients in our definition of supermaps and in the proof of a dilation theorem for them, we devote the next two sections to their explanation.

2.1. *Increasing sequences of normal CP maps*

We now introduce two definitions for sequences of maps in $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ that are analogous to the notion of increasing and bounded sequences of operators. We say that a sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ of elements in $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ is

- *CP-increasing* if $\mathcal{E}_m \preceq \mathcal{E}_n$ whenever $m \leq n$,
- *CP-bounded* if there exists a map $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M},\mathcal{N})$ such that $\mathcal{E}_n \preceq \mathcal{E}$ for all $n \in \mathbb{N}$.

The following result now shows that the notion of lower upper bound can be extended to CP-incresing and CP-bounded sequences in $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$.

Proposition 2.1. *If $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathrm{CP}\,(\mathcal{M},\mathcal{N})$ which is CP-increasing and CP-bounded, then there exists a unique $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M},\mathcal{N})$ such that*

$$\text{wk*-}\lim_n \mathcal{E}_n(A) = \mathcal{E}(A) \quad \forall A \in \mathcal{M}\,.$$

$\mathcal{E}$ has the following property: $\mathcal{E}_n \preceq \mathcal{E}$ for all $n \in \mathbb{N}$, and, if $\mathcal{E}' \in \mathrm{CP}\,(\mathcal{M},\mathcal{N})$ is such that $\mathcal{E}_n \preceq \mathcal{E}'$ for all $n \in \mathbb{N}$, then $\mathcal{E} \preceq \mathcal{E}'$.

If $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and $\mathcal{E}$ are as in the statement of the above proposition, then we write $\mathcal{E}_n \Uparrow \mathcal{E}$.

We now give the most useful application of the previous result. First, note that the simplest example of maps in $\mathrm{CB}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$ is constructed in the following way. Suppose that $\mathcal{M}$ is a von Neumann algebra contained in $\mathcal{L}(\mathcal{H})$. For $E \in \mathcal{L}(\mathcal{H},\mathcal{K})$, $F \in \mathcal{L}(\mathcal{K},\mathcal{H})$, denote by $E \odot_{\mathcal{M}} F$ the linear operator

$$E \odot_{\mathcal{M}} F : \mathcal{M} \to \mathcal{L}(\mathcal{K})\,, \qquad (E \odot_{\mathcal{M}} F)(A) = EAF \quad \forall A \in \mathcal{M}\,.$$

Then it is easy to show that $E \odot_{\mathcal{M}} F \in \mathrm{CB}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$, and, if $E = F^*$, actually $F^* \odot_{\mathcal{M}} F \in \mathrm{CP}\,(\mathcal{M},\mathcal{L}(\mathcal{K}))$. The importance of the elementary maps $E \odot_{\mathcal{M}} F$'s is made clear by the next theorem.

Theorem 2.2 (Kraus theorem). *Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ is a von Neumann algebra. Then, $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ if and only if there exists a finite or countable set $I \subset \mathbb{N}$ and a sequence $\{E_i\}_{i \in I}$ of elements in $\mathcal{L}(\mathcal{K}, \mathcal{H})$ such that the sequence of partial sums $\{\sum_{i \leq n} E_i^* \odot_{\mathcal{M}} E_i\}_{n \in \mathbb{N}}$ converges to $\mathcal{E}$ in $\mathrm{CP}\,(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ in the sense of Proposition 2.1.*

If $\mathcal{E}$ and $\{E_i\}_{i \in I}$ are as in item (2) of the above theorem, the expression $\sum_{i \in I} E_i^* \odot_{\mathcal{M}} E_i$ is the *Kraus form* of $\mathcal{E}$.

Kraus theorem is very important, as it shows that every map in $\mathrm{CB}\,(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ can be decomposed into a (possibly infinite) sum of elementary maps $E_i \odot_{\mathcal{M}} F_i$. Indeed, by Theorem 2.1 we can choose four elements $\mathcal{E}_k \in \mathrm{CP}\,(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ $(k = 0, 1, 2, 3)$ such that $\mathcal{E} = \sum_{k=0}^{3} i^k \mathcal{E}_k$, and by Theorem 2.2 each $\mathcal{E}_k$ can be written in the Kraus form $\mathcal{E}_k = \sum_{i \in I_k} E_i^{(k)*} \odot_{\mathcal{M}} E_i^{(k)}$. It is clear, however, that such decomposition is not unique even if $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ itself.

2.2. *Tensor product of weak*-continuous CB maps*

If $\mathcal{E} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{K}_1)$ and $\mathcal{F} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{K}_2)$ are linear *bounded* maps, their tensor product $\mathcal{E} \otimes \mathcal{F}$ is well defined as a linear map $\mathcal{L}(\mathcal{H}_1) \hat{\otimes} \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{K}_1) \hat{\otimes} \mathcal{L}(\mathcal{K}_2)$. However, unless $\mathcal{H}_1$ and $\mathcal{K}_1$, or alternatively $\mathcal{H}_2$ and $\mathcal{K}_2$, are finite dimensional, in general one can not extend $\mathcal{E} \otimes \mathcal{F}$ to an operator $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$. Weak*-continuous CB maps constitute an important exception to this obstruction, as it is made clear by the following proposition.

Proposition 2.2. *Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$ be von Neumann algebras. Given two maps $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{N}_1)$ and $\mathcal{F} \in \mathrm{CB}\,(\mathcal{M}_2, \mathcal{N}_2)$, there exists a unique map $\mathcal{E} \otimes \mathcal{F} \in \mathrm{CB}\,(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2, \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)$ such that*

$$(\mathcal{E} \otimes \mathcal{F})(A \otimes B) = \mathcal{E}(A) \otimes \mathcal{F}(B) \quad \forall A \in \mathcal{M}_1,\, B \in \mathcal{M}_2\,.$$

If $\mathcal{E}$ and $\mathcal{F}$ are CP, then $\mathcal{E} \otimes \mathcal{F} \in \mathrm{CP}\,(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2, \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)$.

Note that, if $\mathcal{M}_1$ and $\mathcal{N}_1$ (or, equivelently, $\mathcal{M}_2$ and $\mathcal{N}_2$) are finite dimensional, we had already a notion of tensor product at our disposal, i.e. the algebraic tensor product. Indeed, when $\mathcal{M}_1 = M_m(\mathbb{C})$ and $\mathcal{N}_1 = M_n(\mathbb{C})$, the product $\mathcal{E} \otimes \mathcal{F}$ defined in Proposition 2.2 clearly coincides with the algebraic product that we already encountered in the definition of CB and CP maps (see Definition 2.1, where $\mathcal{E} = \mathcal{I}_n$). Moreover, in this case we actually have the equality

$$\mathrm{CB}\,(M_m(\mathbb{C}), M_n(\mathbb{C})) \,\hat{\otimes}\, \mathrm{CB}\,(\mathcal{M}, \mathcal{N}) = \mathrm{CB}\left(\mathcal{M}^{(m)}, \mathcal{N}^{(n)}\right)\,. \tag{2}$$

It is easy to check that the tensor product defined above preserves ordering (i.e. $\mathcal{E}_1 \preceq \mathcal{E}_2$ and $\mathcal{F}_1 \preceq \mathcal{F}_2$ imply $\mathcal{E}_1 \otimes \mathcal{F}_1 \preceq \mathcal{E}_2 \otimes \mathcal{F}_2$) and lower upper bounds (i.e., if $\mathcal{E}_\lambda$, $\mathcal{E}$ and $\mathcal{F}$ are normal CP maps such that $\mathcal{E}_\lambda \Uparrow \mathcal{E}$, then $\mathcal{E}_\lambda \otimes \mathcal{F} \Uparrow \mathcal{E} \otimes \mathcal{F}$).

3. Quantum supermaps

In this section, we introduce the central objects in our study, i.e. the set of linear maps $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$ which mathematically describe the physically admissible transformations of quantum channels. Before giving the precise definition, we need to fix the following terminology.

Definition 3.1. Suppose $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$ are von Neumann algebras. A linear map $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$ is

- *positive* if $\mathsf{S}(\mathcal{E}) \succeq 0$ for all $\mathcal{E} \succeq 0$;
- *completely positive (CP)* if the map
$$\mathsf{I}_n \otimes \mathsf{S} : \mathrm{CB}\left(\mathcal{M}_1^{(n)}, \mathcal{N}_1^{(n)}\right) \to \mathrm{CB}\left(\mathcal{M}_2^{(n)}, \mathcal{N}_2^{(n)}\right)$$
is positive for every $n \in \mathbb{N}$, where I_n is the identity map on the space $\mathrm{CB}(M_n(\mathbb{C}), M_n(\mathbb{C}))$;
- *normal* if $\mathsf{S}(\mathcal{E}_n) \Uparrow \mathsf{S}(\mathcal{E})$ for all sequences $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in $\mathrm{CP}(\mathcal{M}_1, \mathcal{N}_1)$ such that $\mathcal{E}_n \Uparrow \mathcal{E}$.

Note that in the above definition of complete positivity we used the identification $\mathrm{CB}\left(\mathcal{M}^{(n)}, \mathcal{N}^{(n)}\right) = \mathrm{CB}(M_n(\mathbb{C}), M_n(\mathbb{C})) \hat{\otimes} \mathrm{CB}(\mathcal{M}, \mathcal{N})$ of eq. (2).

We are now in position to define quantum supermaps.

Definition 3.2. A *quantum supermap* (or simply, *supermap*) is a normal completely positive linear map $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$.

The convex set of quantum supermaps from $\mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1)$ to $\mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$ will be denoted by $\mathrm{SCP}(\mathcal{M}_1, \mathcal{N}_1; \mathcal{M}_2, \mathcal{N}_2)$. A partial order $\ll$ can be introduced in it as follows: given two maps $\mathsf{S}_1, \mathsf{S}_2 \in \mathrm{SCP}(\mathcal{M}_1, \mathcal{N}_1; \mathcal{M}_2, \mathcal{N}_2)$, we write $\mathsf{S}_1 \ll \mathsf{S}_2$ if $\mathsf{S}_2 - \mathsf{S}_1 \in \mathrm{SCP}(\mathcal{M}_1, \mathcal{N}_1; \mathcal{M}_2, \mathcal{N}_2)$.

We now specialize the definition of quantum supermaps to the following two main cases of interest.

Definition 3.3. A quantum supermap $\mathsf{S} \in \mathrm{SCP}(\mathcal{M}_1, \mathcal{N}_1; \mathcal{M}_2, \mathcal{N}_2)$ is

- *deterministic* if it preserves the set of quantum channels, that is, if $\mathsf{S}(\mathcal{E}) \in \mathrm{CP}_1(\mathcal{M}_2, \mathcal{N}_2)$ for all $\mathcal{E} \in \mathrm{CP}_1(\mathcal{M}_1, \mathcal{N}_1)$;

- *probabilistic* if a deterministic supermap $\mathsf{T} \in \mathrm{SCP}\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_2,\mathcal{N}_2\right)$ exists, such that $\mathsf{S} \ll \mathsf{T}$.

Deterministic supermaps are clearly probabilistic. The subset of deterministic supermaps in $\mathrm{SCP}\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_2,\mathcal{N}_2\right)$ will be labeled by $\mathrm{SCP}_1\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_2,\mathcal{N}_2\right)$.

Obviously, the composition of two quantum supermaps is a supermap: for every $\mathsf{S}_1 \in \mathrm{SCP}\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_2,\mathcal{N}_2\right)$ and $\mathsf{S}_2 \in \mathrm{SCP}\left(\mathcal{M}_2,\mathcal{N}_2;\mathcal{M}_3,\mathcal{N}_3\right)$, we have $\mathsf{S}_2\mathsf{S}_1 \in \mathrm{SCP}\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_3,\mathcal{N}_3\right)$. Similarly, the composition of two probabilistic [resp. deterministic] supermaps is a probabilistic [resp. deterministic] supermap.

We now introduce two examples of supermaps which will play a very important role in the next section.

Example 3.1 (Concatenation). *Given two maps* $\mathcal{A} \in \mathrm{CP}\left(\mathcal{N}_1,\mathcal{N}_2\right)$ *and* $\mathcal{B} \in \mathrm{CP}\left(\mathcal{M}_2,\mathcal{M}_1\right)$*, define the linear map*

$$\begin{aligned}
&\mathsf{C}_{\mathcal{A},\mathcal{B}} : \mathrm{CB}\left(\mathcal{M}_1,\mathcal{N}_1\right) \to \mathrm{CB}\left(\mathcal{M}_2,\mathcal{N}_2\right) ,\\
&\mathsf{C}_{\mathcal{A},\mathcal{B}}(\mathcal{E}) = \mathcal{A}\mathcal{E}\mathcal{B} \quad \forall \mathcal{E} \in \mathrm{CB}\left(\mathcal{M}_1,\mathcal{N}_1\right) .
\end{aligned}$$

Then $\mathsf{C}_{\mathcal{A},\mathcal{B}} \in \mathrm{SCP}\left(\mathcal{M}_1,\mathcal{N}_1;\mathcal{M}_2,\mathcal{N}_2\right)$*. Moreover, if* $\mathcal{A}$ *and* $\mathcal{B}$ *are quantum channels, then* $\mathsf{C}_{\mathcal{A},\mathcal{B}}$ *is deterministic.*

Example 3.2 (Amplification). *Suppose* $\mathcal{V}$ *is a Hilbert space, and define the linear map*

$$\begin{aligned}
&\Pi_{\mathcal{V}} : \mathrm{CB}\left(\mathcal{M},\mathcal{N}\right) \to \mathrm{CB}\left(\mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{V}),\mathcal{N}\bar{\otimes}\mathcal{L}(\mathcal{V})\right) ,\\
&\Pi_{\mathcal{V}}(\mathcal{E}) = \mathcal{E}\otimes\mathcal{I}_{\mathcal{V}} \quad \forall \mathcal{E} \in \mathrm{CB}\left(\mathcal{M},\mathcal{N}\right) ,
\end{aligned}$$

where we recall that $\mathcal{I}_{\mathcal{V}} := \mathcal{I}_{\mathcal{L}(\mathcal{V})}$ *(cf. Proposition 2.2 for the definition of the tensor product). Then the map* $\Pi_{\mathcal{V}}$ *is a deterministic supermap, that is,* $\Pi_{\mathcal{V}} \in \mathrm{SCP}_1\left(\mathcal{M},\mathcal{N};\mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{V}),\mathcal{N}\bar{\otimes}\mathcal{L}(\mathcal{V})\right)$.

The main result in the next two sections is that every deterministic or probabilistic supermap in $\mathrm{SCP}\left(\mathcal{M}_1,\mathcal{L}(\mathcal{K}_1);\mathcal{M}_2,\mathcal{L}(\mathcal{K}_2)\right)$ is the composition of an amplification followed by a concatenation (Theorems 4.1 and 4.2 below).

4. Dilation of deterministic and probabilistic supermaps

Our central result is the following dilation theorem for deterministic supermaps.

Theorem 4.1 (Dilation of deterministic supermaps). *Suppose* $\mathcal{M}_1$, $\mathcal{M}_2$ *are von Neumann algebras. A linear map* $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ *is a deterministic supermap if and only if there exists a triple* $(\mathcal{V}, V, \mathcal{F})$*, where*

- $\mathcal{V}$ *is a Hilbert space*
- $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$ *is an isometry*
- $\mathcal{F}$ *is a quantum channel in* $\mathrm{CP}_1(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$

such that

$$[\mathsf{S}(\mathcal{E})](A) = V^* [(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})\mathcal{F}(A)] V \quad \forall \mathcal{E} \in \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)), \, A \in \mathcal{M}_2 . \quad (3)$$

The triple $(\mathcal{V}, V, \mathcal{F})$ *can always be chosen in a way that*

$$\mathcal{V} = \overline{\mathrm{span}} \, \{(u^* \otimes I_{\mathcal{V}})Vv \mid u \in \mathcal{K}_1 \, , \, v \in \mathcal{K}_2\} \, . \quad (4)$$

We remark that in eq. (4) the adjoint u^* of $u \in \mathcal{K}_1$ is the linear functional $u^* : w \mapsto \langle u, w \rangle$ on $\mathcal{K}_1$.

Note that, if we define the quantum channel $\mathcal{A} := V^* \odot_{\mathcal{L}(\mathcal{K}_1 \otimes \mathcal{V})} V$, then eq. (3) is equivalent to

$$\mathsf{S} = \mathsf{C}_{\mathcal{A},\mathcal{F}} \Pi_{\mathcal{V}} \, ,$$

where $\mathsf{C}_{\mathcal{A},\mathcal{F}}$ and $\Pi_{\mathcal{V}}$ are the concatenation and amplification supermaps defined in Examples 3.1 and 3.2. In particular, we see that, if a linear map $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ is defined as in eq. (3), then $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ by the composition property of deterministic supermaps. The converse statement is more difficult to be shown, and a sketch of its proof will be provided in the next subsection.

Definition 4.1. If a Hilbert space $\mathcal{V}$, an isometry $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$, and a quantum channel $\mathcal{F} \in \mathrm{CP}_1(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$ are such that eq. (3) holds, then we say that the triple $(\mathcal{V}, V, \mathcal{F})$ is a *dilation* of the supermap S. If also eq. (4) holds, then we say that the dilation $(\mathcal{V}, V, \mathcal{F})$ is *minimal.*

The importance of the minimality property is highlighted by the following fact.

Proposition 4.1. *Let* $(\mathcal{V}, V, \mathcal{F})$ *and* $(\mathcal{V}', V', \mathcal{F}')$ *be two dilations of the deterministic supermap* $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$*. If* $(\mathcal{V}, V, \mathcal{F})$ *is minimal, then there exists a unique isometry* $W : \mathcal{V} \to \mathcal{V}'$ *such that* $V' = (I_{\mathcal{K}_1} \otimes W)V$ *and* $\mathcal{F}(A) = (I_{\mathcal{M}_1} \otimes W^*)\mathcal{F}'(A)(I_{\mathcal{M}_1} \otimes W)$ *for all* $A \in \mathcal{M}_2$*. Moreover, if also the dilation* $(\mathcal{V}', V', \mathcal{F}')$ *is minimal, then the isometry* W *is actually unitary.*

Remark 4.1. Suppose $\mathcal{M}_1 = \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{M}_2 \subset \mathcal{L}(\mathcal{H}_2)$. In this case, a linear map $\mathsf{S} : \mathrm{CB}\,(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{K}_1)) \to \mathrm{CB}\,(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ is a deterministic supermap if and only if there exist two separable Hilbert spaces $\mathcal{V}$,$\mathcal{U}$ and two isometries $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$, $U : \mathcal{H}_1 \otimes \mathcal{V} \to \mathcal{H}_2 \otimes \mathcal{U}$ such that

$$[\mathsf{S}(\mathcal{E})](A) = V^* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})(U^*(A \otimes I_{\mathcal{U}})U) \right] V \tag{5}$$

for all $\mathcal{E} \in \mathrm{CB}\,(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{K}_1))$ and $A \in \mathcal{M}_2$. Indeed, by Stinespring theorem (Theorem 4.3 p. 165 in Ref. 28 and the discussion following it) every quantum channel $\mathcal{F} \in \mathrm{CP}_1\,(\mathcal{M}_2, \mathcal{L}(\mathcal{H}_1)\bar{\otimes}\mathcal{L}(\mathcal{V})) = \mathrm{CP}_1\,(\mathcal{M}_2, \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{V}))$ can be written as

$$\mathcal{F}(A) = U^*(A \otimes I_{\mathcal{U}})U \quad \forall A \in \mathcal{M}_2$$

for some separable Hilbert space $\mathcal{U}$ and some isometry $U : \mathcal{H}_1 \otimes \mathcal{V} \to \mathcal{H}_2 \otimes \mathcal{U}$. Eq. (5) then follows by eq. (3). Note that in this way we recover Theorem 5 of Ref. 19 as a particular case of Theorem 4.1 above.

Remark 4.2. As anticipated in the Introduction, eq. (3) is the desired generalization of the analogous finite dimensional result in Refs. 1,2. The physical interpretation of the dilation of deterministic supermaps is clear in the Schrödinger picture: indeed, turning eq. (3) into its predual, we obtain

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \mathcal{F}_* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*(V \rho V^*) \right]$$

for all ρ in the space $\mathcal{T}(\mathcal{K}_2)$ of trace class operators on $\mathcal{K}_2$ and $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$. If $\mathcal{M}_i = \mathcal{L}(\mathcal{H}_i)$, take the Stinespring dilation $\mathcal{F}(A) = U^*(A \otimes I_{\mathcal{U}})U$ of $\mathcal{F}$. The last equation then rewrites

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \mathrm{tr}_{\mathcal{U}} \left\{ U \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*(V \rho V^*) \right] U^* \right\}$$

where $\mathrm{tr}_{\mathcal{U}}$ denotes the partial trace over $\mathcal{U}$. If ρ is a quantum state (i.e. $\rho \geq 0$ and $\mathrm{tr}\,(\rho) = 1$), this means that the quantum system with Hilbert space $\mathcal{K}_2$ first undergoes the invertible evolution V, then the dilated quantum channel $(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*$, and finally the invertible evolution U, after which the ancillary system with Hilbert space $\mathcal{U}$ is discarded. It is interesting to note that the same kind of sequential composition of invertible evolutions also appears in a very different context: the reconstruction of quantum stochastic processes from correlation kernels.[29–31] That context is very different from the present framework of higher-order maps, and it is a remarkable feature of Theorem 4.1 that any deterministic supermap on the space of quantum operations can be achieved through a two-step sequence of invertible evolutions.

Theorem 4.1 contains as a special case the Stinespring dilation of quantum channels. This fact is illustrated in the following example.

Example 4.1 (Stinespring theorem). *Suppose that* $\mathcal{M}_1 = \mathcal{M}_2 = \mathbb{C}$, *the trivial von Neumann algebra. In this case we have the identification* $\mathrm{CB}(\mathbb{C}, \mathcal{L}(\mathcal{K}_i)) = \mathcal{L}(\mathcal{K}_i)$. *Precisely, the element* $\mathcal{E} \in \mathrm{CB}(\mathbb{C}, \mathcal{K}_i)$ *is identified with the operator* $A_{\mathcal{E}} = \mathcal{E}(1) \in \mathcal{L}(\mathcal{K}_i)$. *Moreover, we clearly have* $\mathrm{CP}_1(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V})) = \{I_{\mathcal{V}}\}$, *hence eq.* (3) *reads*

$$[\mathsf{S}(\mathcal{E})](1) = V^*(A_{\mathcal{E}} \otimes I_{\mathcal{V}})V ,$$

which is just Stinespring dilation for normal CP maps. A linear map $\mathsf{S} : \mathcal{L}(\mathcal{K}_1) \to \mathcal{L}(\mathcal{K}_2)$ *is thus in* $\mathrm{SCP}_1(\mathbb{C}, \mathcal{L}(\mathcal{K}_1); \mathbb{C}, \mathcal{L}(\mathcal{K}_2))$ *if and only if it is a unital normal CP map, i.e. a quantum channel.*

The dilation theorem for deterministic supermaps can be generalized to probabilistic supermaps. In this case, the following theorem provides an analog of the Radon-Nikodym theorem for CP maps (compare with Refs. 32, 33, and see also Ref. 34 for the particular case of quantum operations).

Theorem 4.2 (Radon-Nikodym theorem for supermaps). *Suppose* $\mathsf{S} \in \mathrm{SCP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ *and let* $(\mathcal{V}, V, \mathcal{F})$ *be its minimal dilation. If* $\mathsf{T} \in \mathrm{SCP}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ *is such that* $\mathsf{T} \ll \mathsf{S}$, *then there exists a unique element* $\mathcal{G} \in \mathrm{CP}(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$ *with* $\mathcal{G} \preceq \mathcal{F}$ *and such that*

$$[\mathsf{T}(\mathcal{E})](A) = V^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})\mathcal{G}(A)]V \quad \forall \mathcal{E} \in \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)),\ A \in \mathcal{M}_2 . \tag{6}$$

Definition 4.2. With the notations of Theorem 4.2, the map $\mathcal{G} \in \mathrm{CP}(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$ defined by eq. (6) is the *Radon-Nikodym derivative* of the supermap T with respect to S.

4.1. *Sketch of the proof of Theorem 4.1*

Here we provide a sketch of the proof of our central dilation Theorem 4.1. The interested reader is referred to Refs. 19,20 for the details.

In the following, we will restrict ourselves to the simplified case in which the deterministic supermap S belongs to the set $\mathrm{SCP}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}); \mathcal{N}, \mathcal{L}(\mathcal{K}))$, i.e. assume $\mathcal{M}_1 = \mathcal{L}(\mathcal{K}_1)$ in the notations of Theorem 4.1. The proof can be divided into several steps.

(1) Each supermap $\mathsf{S} \in \mathrm{SCP}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}); \mathcal{N}, \mathcal{L}(\mathcal{K}))$ defines a sesquilinear form $\langle \cdot, \cdot \rangle_1$ on the algebraic tensor product $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{N} \hat{\otimes} \mathcal{K}$ as follows

$$\langle E_1 \otimes A_1 \otimes v_1, E_2 \otimes A_2 \otimes v_2 \rangle_1 := \langle v_1, [\mathsf{S}(E_1^* \odot_{\mathcal{M}_1} E_2)](A_1^* A_2) v_2 \rangle .$$

It is not difficult to show that complete positivity of S implies that the form $\langle \cdot, \cdot \rangle_1$ is positive semidefinite. If $\mathcal{R} \subset \mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{N} \hat{\otimes} \mathcal{K}$ is the radical of the form $\langle \cdot, \cdot \rangle_1$, the quotient space $\mathcal{L}(\mathcal{H}) \hat{\otimes} \mathcal{N} \hat{\otimes} \mathcal{K} / \mathcal{R}$ can then be completed to a

Hilbert space, say $\hat{\mathcal{U}}_1$. We denote by $\langle\cdot,\cdot\rangle_1$ the resulting scalar product in $\hat{\mathcal{U}}_1$.

(2) We can also use $\langle\cdot,\cdot\rangle_1$ to introduce a second positive semidefinite sesquilinear form $\langle\cdot,\cdot\rangle_2$, this time defined on the algebraic tensor product $\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}$ and given by

$$\langle E_1 \otimes v_1, E_2 \otimes v_2\rangle_2 := \langle E_1 \otimes I_{\mathcal{N}} \otimes v_1, E_2 \otimes I_{\mathcal{N}} \otimes v_2\rangle_1 .$$

As before, if $\mathcal{R}' \subset \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}$ is the radical of $\langle\cdot,\cdot\rangle_2$, we denote by $\hat{\mathcal{U}}_2$ the Hilbert space completion of the quotient $\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}/\mathcal{R}'$, and let $\langle\cdot,\cdot\rangle_2$ be the scalar product extended to $\hat{\mathcal{U}}_2$. It can be proven that the Hilbert space $\hat{\mathcal{U}}_2$ is separable.

(3) Now, we define two linear maps

$$\begin{aligned} &U_1 : \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K} \to \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{N}\hat{\otimes}\mathcal{K} \qquad && U_1(E\otimes v) = E \otimes I_{\mathcal{N}} \otimes v \\ &U_2 : \mathcal{K} \to \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K} && U_2 v = I_{\mathcal{H}} \otimes v . \end{aligned}$$

It is easy to verify by definitions that U_1 and U_2 extend to isometries $U_1 : \hat{\mathcal{U}}_2 \to \hat{\mathcal{U}}_1$ and $U_2 : \mathcal{K} \to \hat{\mathcal{U}}_2$, respectively.

(4) For all $B \in \mathcal{N}$, we introduce the linear operator $\pi_1(B)$ on $\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{N}\hat{\otimes}\mathcal{K}$, defined by

$$[\pi_1(B)](E \otimes A \otimes v) = E \otimes BA \otimes v$$

for all $E \in \mathcal{L}(\mathcal{H})$, $A \in \mathcal{N}$ and $v \in \mathcal{K}$. Using again the definitions, it is easy to show that π_1 extends to a normal unital $*$-homomorphism of $\mathcal{N}$ into $\mathcal{L}(\hat{\mathcal{U}}_1)$.

(5) For all $F \in \mathcal{L}(\mathcal{H})$, we introduce the linear operator $\pi_2(F)$ on $\mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{K}$, defined by

$$[\pi_2(F)](E \otimes v) = FE \otimes v$$

for all $E \in \mathcal{L}(\mathcal{H})$ and $v \in \mathcal{K}$. It can be shown that also π_2 extends to a normal unital $*$-homomorphism of $\mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\hat{\mathcal{U}}_2)$. However, we remark that in this case the proof is more involved than in step (4), and makes essential use of the fact that the supermap S is deterministic (see Lemma 4.3 in Ref. 20) and normal.

(6) By Lemma 2.2 p. 139 in Ref. 3, separability of $\hat{\mathcal{U}}_2$ and normality of π_2 imply that there exists a (separable) Hilbert space $\mathcal{V}$ such that $\hat{\mathcal{U}}_2 = \mathcal{H} \otimes \mathcal{V}$ and $\pi_2(F) = F \otimes I_{\mathcal{V}}$ for all $F \in \mathcal{L}(\mathcal{H})$. Note that, if $E \in \mathcal{L}(\mathcal{H})$, $A \in \mathcal{N}$ and $v \in \mathcal{K}$, then by an immediate application of definitions we have

$$\pi_1(A)U_1\pi_2(E)U_2 v = E \otimes A \otimes v \quad \text{as an element of } \mathcal{L}(\mathcal{H})\hat{\otimes}\mathcal{N}\hat{\otimes}\mathcal{K}.$$

(7) We define a linear map $\mathcal{F} : \mathcal{N} \to \mathcal{L}(\hat{\mathcal{U}}_2) = \mathcal{L}(\mathcal{H} \otimes \mathcal{V})$, given by

$$\mathcal{F}(A) := U_1^* \pi_1(A) U_1 \quad \forall A \in \mathcal{N} .$$

Clearly, $\mathcal{F}$ is a unital CP map. By normality of the representation π_1, it follows that actually $\mathcal{F} \in \mathrm{CP}_1 \left(\mathcal{N}, \mathcal{L}(\mathcal{H} \otimes \mathcal{V}) \right)$.

(8) At this point, we are in position to prove eq. (3) for elementary CP maps. Indeed, if $E \in \mathcal{L}(\mathcal{H})$, $A \in \mathcal{N}$ and $v, w \in \mathcal{K}$, then we have, for $\mathcal{E} = E^* \odot_{\mathcal{L}(\mathcal{H})} E$,

$$\begin{aligned} \langle v, [\mathsf{S}(\mathcal{E})] (A) w \rangle &= \langle E \otimes I_{\mathcal{N}} \otimes v, E \otimes A \otimes w \rangle_1 \\ &= \langle U_1 \pi_2(E) U_2 v, \pi_1(A) U_1 \pi_2(E) U_2 w \rangle_1 \\ &= \langle \pi_2(E) U_2 v, \mathcal{F}(A) \pi_2(E) U_2 w \rangle_2 \\ &= \langle v, U_2^* (E^* \otimes I_{\mathcal{V}}) \mathcal{F}(A) (E \otimes I_{\mathcal{V}}) U_2 w \rangle \\ &= \langle v, U_2^* [(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}) \mathcal{F}(A)] U_2 w \rangle \ . \end{aligned}$$

Setting $V := U_2$, we then obtain

$$[\mathsf{S}(\mathcal{E})] (A) = V^* [(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}) \mathcal{F}(A)] V \quad \forall A \in \mathcal{N} ,$$

i.e. eq. (3) in the special case $\mathcal{E} = E^* \odot_{\mathcal{L}(\mathcal{H})} E$.

(9) By Kraus Theorem 2.2, eq. (3) for generic $\mathcal{E} \in \mathrm{CP} \left(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}) \right)$ then follows from step (8) using normality of S and of the amplification supermap $\mathsf{\Pi}_{\mathcal{U}} : \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{I}_{\mathcal{U}}$. Finally, linearity and Theorem 2.1 extend the equality to all $\mathcal{E} \in \mathrm{CB} \left(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}) \right)$. This concludes the proof of Theorem 4.1.

5. An application of Theorem 4.1: transforming a quantum measurement into a quantum channel

For simplicity we consider here quantum measurements with a countable set of outcomes, denoted by X. In the algebraic language, a measurement on the quantum system with Hilbert space $\mathcal{K}_1$ and with outcomes in X is described by a quantum channel $\mathcal{E} \in \mathrm{CP}_1 \left(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1) \right)$, where $\mathcal{M}_1 \equiv \ell^\infty(X)$ is the von Neumann algebra of the bounded complex functions (i.e. sequences) on X with uniform norm $\left\| f \right\|_\infty := \sup_{i \in X} |f_i|$. The channel $\mathcal{E}$ maps the function $f \in \ell^\infty(X)$ into the operator

$$\mathcal{E}(f) = \sum_{i \in X} f_i \, P_i \in \mathcal{L}(\mathcal{K}_1) \, , \tag{7}$$

where each P_i is a non-negative operator in $\mathcal{L}(\mathcal{K}_1)$ and $\sum_{i \in X} P_i = I_{\mathcal{K}_1}$. Note that the map $i \mapsto P_i$ is a normalized *positive operator valued measure (POVM)* based on the discrete space X and with values in $\mathcal{L}(\mathcal{K}_1)$.

Actually, eq. (7) allows us to identify the convex set of measurements $\mathrm{CP}_1(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))$ with the set of *all* normalized $\mathcal{L}(\mathcal{K}_1)$-valued POVMs on X.[a]

The probability of obtaining the outcome $i \in X$ when the measurement is performed on a system prepared in the quantum state $\rho \in \mathcal{T}(\mathcal{K}_1) = \mathcal{L}(\mathcal{K}_1)_*$ is given by the Born rule

$$p_i = \mathrm{tr}\,(\rho P_i)\ ,$$

and the expectation value of the function $f \in \ell^\infty(X)$ with respect to the probability distribution p is given by

$$\mathbb{E}_p(f) := \sum_{i\in X} p_i f_i = \mathrm{tr}\,[\rho\mathcal{E}(f)]\ .$$

The above equation allows us to interpret the channel $\mathcal{E}$ as an *operator valued expectation* (see e.g. Ref. 35).

Now, consider the deterministic supermaps sending quantum measurements in $\mathrm{CP}(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))$ to quantum operations in $\mathrm{CP}(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$, where $\mathcal{M}_2 \equiv \mathcal{L}(\mathcal{H}_2)$. In this case, our dilation Theorem 4.1 (in the predual form of Remark 4.2) states that every deterministic supermap $\mathsf{S} : \mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1)) \to \mathrm{CB}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{K}_2))$ is of the form

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \mathcal{F}_*[(\mathcal{E}\otimes\mathcal{I}_\mathcal{V})_*(V\rho V^*)] \quad \forall\mathcal{E}\in\mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))\,,\ \rho\in\mathcal{T}(\mathcal{K}_2) \tag{8}$$

where $\mathcal{V}$ is a Hilbert space, $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$ is an isometry, and $\mathcal{F} \in \mathrm{CP}_1(\mathcal{L}(\mathcal{H}_2), \ell^\infty(X)\bar{\otimes}\mathcal{L}(\mathcal{V}))$ is a quantum channel. In our case, we have the identification

$$\ell^\infty(X)\bar{\otimes}\mathcal{L}(\mathcal{V}) \simeq \ell^\infty(X;\mathcal{L}(\mathcal{V}))\,,$$

where $\ell^\infty(X;\mathcal{L}(\mathcal{V}))$ is the von Neumann algebra of the bounded $\mathcal{L}(\mathcal{V})$-valued functions on X. Its predual space is

$$(\ell^\infty(X)\bar{\otimes}\mathcal{L}(\mathcal{V}))_* \simeq \ell^1(X;\mathcal{T}(\mathcal{V}))\,,$$

i.e. the space of norm-summable sequences with index in X and values in the Banach space of the trace class operators on $\mathcal{V}$ (see Theorem 1.22.13 in Ref. 25). In the Schrödinger picture, the channel $\mathcal{F}_*$ can be realized by first reading the classical information carried by the system with algebra

[a]Indeed, by commutativity of $\ell^\infty(X)$ the set $\mathrm{CP}_1(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))$ coincides with the set of all normalized weak*-continuous *positive* maps from $\ell^\infty(X)$ into $\mathcal{L}(\mathcal{K}_1)$ (Theorem 3.11 in Ref. 26). The latter set is just the set of all normalized $\mathcal{L}(\mathcal{K}_1)$-valued POVMs on X, the identification being the one given in eq. (7).

$\ell^\infty(X)$ and, conditionally to the value $i \in X$, by performing the quantum channel $\mathcal{F}_{i\,*} : \mathcal{T}(\mathcal{V}) \to \mathcal{T}(\mathcal{H}_2)$ given by

$$\mathcal{F}_{i\,*}(\sigma) = \mathcal{F}_*(\delta_i\,\sigma) \quad \forall \sigma \in \mathcal{T}(\mathcal{V}) \ ,$$

where $\delta_i\,\sigma \in \ell^1(X;\mathcal{T}(\mathcal{V}))$ is the sequence $(\delta_i\,\sigma)_k = \delta_{ik}\,\sigma\ \forall k \in X$ (δ_{ik} is just Kronecker delta). Indeed, in this way eq. (8) can be rewritten

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \sum_{i\in X} \mathcal{F}_{i\,*}[(\mathcal{E}\otimes\mathcal{I}_{\mathcal{V}})_*(V\rho V^*)_i]\,.$$

In other words, Theorem 4.1 states that the most general transformation of a quantum measurement on $\mathcal{K}_1$ into a quantum channel from states on $\mathcal{K}_2$ to states on $\mathcal{H}_2$ can be realized by

(1) applying an invertible dynamics (the isometry V) that transforms the input system $\mathcal{K}_2$ into the composite system $\mathcal{K}_1 \otimes \mathcal{V}$, where $\mathcal{V}$ is an ancillary system;
(2) performing the given measurement $\mathcal{E}$ on $\mathcal{K}_1$, thus obtaining the outcome $i \in X$;
(3) conditionally to the outcome $i \in X$, applying a physical transformation (the channel $\mathcal{F}_{i\,*}$) on the ancillary system $\mathcal{V}$, thus converting it into the output system $\mathcal{H}_2$.

6. Superinstruments

Quantum superinstruments describe measurement processes where the measured object is not a quantum system, as in ordinary instruments, but rather a quantum device. While ordinary quantum instruments are defined as measures with values in the set of quantum operations (see Ref. 22, and also Ref. 3 for a more complete exposition), quantum superinstruments are defined as probability measures with values in the set of quantum supermaps.

Definition 6.1. Let Ω be a measurable space with σ-algebra $\sigma(\Omega)$ and let S be a map from $\sigma(\Omega)$ to $\mathrm{SCP}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$, sending the measurable subset $B \in \sigma(\Omega)$ to the quantum supermap $\mathsf{S}_B \in \mathrm{SCP}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$. We say that S is a *quantum superinstrument* if it satisfies the following properties:

(i) S_Ω is deterministic;
(ii) if $n \in \mathbb{N}\cup\{\infty\}$ and $B = \bigcup_{i=1}^n B_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, then $\mathsf{S}_B = \sum_{i=1}^n \mathsf{S}_{B_i}$, where if $n = \infty$ convergence of the series is understood

in the following sense:

$$[\mathsf{S}_B(\mathcal{E})](A) = \text{wk*-}\lim_k \sum_{i=1}^{k} [\mathsf{S}_{B_i}(\mathcal{E})](A)$$

for all $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$ and $A \in \mathcal{M}_2$.

We will briefly see that every quantum superinstrument is associated to an ordinary quantum instrument in an unique way. Before giving the precise statement, we recall the notion of quantum instrument, which is central in the statistical description of quantum measurements:

Definition 6.2. A map $\mathcal{J} : \sigma(\Omega) \to \mathrm{CP}\,(\mathcal{M}, \mathcal{N})$ is a *quantum instrument* if it satisfies the following properties:

(i) $\mathcal{J}_\Omega$ is a quantum channel;
(ii) if $n \in \mathbb{N} \cup \{\infty\}$ and $B = \bigcup_{i=1}^n B_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, then $\mathcal{J}_B = \sum_{i=1}^n \mathcal{J}_{B_i}$, where if $n = \infty$ convergence of the series is understood in the following sense:

$$\mathcal{J}_B(A) = \text{wk*-}\lim_k \sum_{i=1}^{k} \mathcal{J}_{B_i}(A) \quad \forall A \in \mathcal{M}\,.$$

With an easy application of Radon-Nikodym Theorem 4.2, one can then prove the following dilation theorem for quantum superinstruments.

Theorem 6.1 (Dilation of quantum superinstruments). *Suppose that* $\mathsf{S} : \sigma(\Omega) \to \mathrm{SCP}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1); \mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ *is a quantum superinstrument and let* $(\mathcal{V}, V, \mathcal{F})$ *be the minimal dilation of the deterministic supermap* S_Ω*. Then there exists a unique quantum instrument* $\mathcal{J} : \sigma(\Omega) \to \mathrm{CP}\,(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$ *such that*

$$[\mathsf{S}_B(\mathcal{E})](A) = V^*[(\mathcal{E} \otimes \mathcal{I}_\mathcal{V})\mathcal{J}_B(A)]V \quad \forall \mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))\,,\ A \in \mathcal{M}_2 \quad (9)$$

for all $B \in \sigma(\Omega)$.

The physical interpretation of the dilation of quantum superinstruments is clear in the Schrödinger picture. Indeed, taking the predual of eq. (9), we have for all $\rho \in \mathcal{T}(\mathcal{K}_2)$ and $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$

$$[\mathsf{S}_B(\mathcal{E})]_*(\rho) = \mathcal{J}_{B\,*}\left[(\mathcal{E} \otimes \mathcal{I}_\mathcal{V})_*(V\rho V^*)\right]\,.$$

This means that a quantum state ρ first is coupled with an ancillary system with Hilbert space $\mathcal{V}$ and the overall system undergoes the invertible evolution V; then the system is transformed by means of the quantum channel $\mathcal{E}$, while nothing is done on the ancilla; finally, the quantum measurement $\mathcal{J}$ is performed on the system + ancilla, and after that the ancilla is discarded.

7. Application of Theorem 6.1: Measuring a measurement

Suppose that we want to characterize some property of a quantum measuring device on a system with Hilbert space $\mathcal{K}_1$: for example, we may have a device performing a projective measurement on an unknown orthonormal basis, and want to find out the basis. In this case the set of possible answers to our question is thus the set of all orthonormal bases. In a more abstract setting, the possible outcomes will constitute a measure space Ω with σ-algebra $\sigma(\Omega)$. This includes also the case of full tomography of the measurement device,[36–39] in which the outcomes in Ω label all possible measurements.

The mathematical object describing our task will be a superinstrument taking the given measurement as input and yielding an outcome in the set $B \in \sigma(\Omega)$ with some probability. In the algebraic framework, we will describe the input measurement as a quantum channel $\mathcal{E} \in \mathrm{CP}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$, where $\mathcal{M}_1 \equiv \ell^\infty(X)$ is the algebra of the complex bounded functions on X (see the discussion in Section 5).

7.1. *Outcome statistics for a measurement on a measuring device*

If we only care about the outcomes in Ω and their statistical distribution, then the output of the superinstrument will be trivial, that is $\mathcal{M}_2 \equiv \mathcal{L}(\mathcal{K}_2) \equiv \mathbb{C}$. In this case, Theorem 6.1 states that every superinstrument $\mathsf{S} : \sigma(\Omega) \to \mathrm{SCP}\,(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1); \mathbb{C}, \mathbb{C})$ will be of the form

$$\mathsf{S}_B(\mathcal{E}) = \langle v, (\mathcal{E} \otimes \mathcal{I}_\mathcal{V})(\mathcal{J}_B) v \rangle \quad \forall \mathcal{E} \in \mathrm{CB}\,(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))\,,\ B \in \sigma(\Omega)\,,$$

where $\mathcal{V}$ is an ancillary Hilbert space, $v \in \mathcal{K}_1 \otimes \mathcal{V}$ is a unit vector, and $\mathcal{J} : \sigma(\Omega) \to \mathrm{CP}\,(\mathbb{C}, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V})) \simeq \ell^\infty(X; \mathcal{L}(\mathcal{V}))$ is just a weak*-countably additive positive measure on Ω with values in $\ell^\infty(X; \mathcal{L}(\mathcal{V}))$, satisfying $(\mathcal{J}_\Omega)_i = I_\mathcal{V}\ \forall i \in X$. Note that in this case each supermap S_B is actually a linear map $\mathrm{CB}\,(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1)) \to \mathbb{C}$, and, if $\mathcal{E}$ is a quantum channel, the map $B \mapsto \mathsf{S}_B(\mathcal{E})$ is a probability measure on Ω. In the Schrödinger picture

$$\mathsf{S}_B(\mathcal{E}) = [\mathcal{J}_{B\,*}(\mathcal{E} \otimes \mathcal{I}_\mathcal{V})_*](\omega_v)\,, \tag{10}$$

where ω_v is the state in $\mathcal{T}\,(\mathcal{K}_1 \otimes \mathcal{V})$ given by $\omega_v(A) := \langle v, Av \rangle\ \forall A \in \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{V})$. Note that $\mathcal{J}_{B\,*} : \ell^1(X; \mathcal{T}\,(\mathcal{V})) \to \mathbb{C}$. Thus, if for all $i \in X$ we define the following normalized POVM on Ω

$$Q_i : \sigma(\Omega) \to \mathcal{L}(\mathcal{V})\,, \qquad Q_{i,B} := (\mathcal{J}_B)_i\,,$$

then we have

$$\mathcal{J}_{B\,*}(\delta_i\,\sigma) = \mathrm{tr}\,(\sigma Q_{i,B}) \quad \forall \sigma \in \mathcal{T}\,(\mathcal{V})$$

and eq. (10) becomes

$$\mathsf{S}_B(\mathcal{E}) = \sum_{i\in X} \mathrm{tr}\,[Q_{i,B}(\mathcal{E}\otimes\mathcal{I}_{\mathcal{V}})_*(\omega_v)_i]\ ,$$

which shows that, conditionally on the classical information $i \in X$, we just perform a measurement of the normalized POVM Q_i on the states in $\mathcal{T}\,(\mathcal{V})$. In other words, Theorem 6.1 claims that the most general way to extract information about a measuring device on system $\mathcal{K}_1$ consists in

(1) preparing a pure bipartite state ω_v in $\mathcal{K}_1 \otimes \mathcal{V}$;
(2) performing the given measurement $\mathcal{E}$ on $\mathcal{K}_1$, thus obtaining the outcome $i \in X$;
(3) conditionally on the outcome $i \in X$, performing a measurement (the POVM Q_i) on the ancillary system $\mathcal{V}$, thus obtaining an outcome in Ω.

Note that the choice of the POVM Q_i depends in general on the outcome of the first measurement $\mathcal{E}$.

7.2. *Tranformations of measuring devices induced by a higher-order measurement*

In a quantum measurement it is often interesting to consider not only the statistics of the outcomes, but also how the measured object changes due to the measurement process. For example, in the case of ordinary quantum measurements, one is interested in studying the state reduction due to the occurrence of particular measurement outcomes We can ask the same question in the case of higher-order measurements on quantum devices: for example, we can imagine a measurement process where a measuring device is tested, producing outcomes in Ω, and transformed into a new measuring device. This situation is described mathematically by a quantum superinstrument with outcomes in Ω, sending measurements in $\mathrm{CP}\,(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$ to measurements in $\mathrm{CP}\,(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$, where $\mathcal{M}_1 \equiv \ell^\infty(X)$ and $\mathcal{M}_2 \equiv \ell^\infty(Y)$ for some countable sets X and Y.

In this case, Theorem 6.1 states that every superinstrument $\mathsf{S} : \sigma(\Omega) \to \mathrm{SCP}\,(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1); \ell^\infty(Y), \mathcal{L}(\mathcal{K}_2))$ is of the form

$$[\mathsf{S}_B(\mathcal{E})](f) = V^*[(\mathcal{E}\otimes\mathcal{I}_{\mathcal{V}})\mathcal{J}_B(f)]V \quad \forall \mathcal{E} \in \mathrm{CB}\,(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1))\,,\ f \in \ell^\infty(Y)$$

for all $B \in \sigma(\Omega)$, where $\mathcal{V}$ is an ancillary Hilbert space, $V \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1 \otimes \mathcal{V})$ is an isometry, and $\mathcal{J} : \sigma(\Omega) \to \mathrm{CP}\,(\ell^\infty(Y), \ell^\infty(X; \mathcal{L}(\mathcal{V})))$ is an instrument.

Note that, by commutativity of $\ell^\infty(Y)$, the set $\mathrm{CP}\,(\ell^\infty(Y), \ell^\infty(X;\mathcal{L}(\mathcal{V})))$ is actually the set of weak*-continuous *positive* maps from $\ell^\infty(Y)$ into $\ell^\infty(X;\mathcal{L}(\mathcal{V}))$. If for all $i \in X$ we define the positive map

$$\mathcal{J}_{i,B} : \ell^\infty(Y) \to \mathcal{L}(\mathcal{V})\,, \qquad \mathcal{J}_{i,B}(f) := \mathcal{J}_B(f)_i\,,$$

then the mapping $\mathcal{J}_i : \sigma(\Omega) \to \mathrm{CP}\,(\ell^\infty(Y), \mathcal{L}(\mathcal{V}))$ is an instrument, with preduals

$$\mathcal{J}_{i,B\,*} : \mathcal{T}\,(\mathcal{V}) \to \ell^1(Y)\,, \qquad \mathcal{J}_{i,B\,*}(\sigma) = \mathcal{J}_{B\,*}(\delta_i\,\sigma)$$

for all $B \in \sigma(\Omega)$. From the relation

$$[\mathsf{S}_B(\mathcal{E})]_*(\rho) = [\mathcal{J}_{B\,*}(\mathcal{E}\otimes\mathcal{I}_\mathcal{V})_*](V\rho V^*) = \sum_{i\in X} \mathcal{J}_{i,B\,*}[(\mathcal{E}\otimes\mathcal{I}_\mathcal{V})_*(V\rho V^*)_i]\,,$$

holding for all states $\rho \in \mathcal{T}\,(\mathcal{K}_2)$, we then see that the most general measurement on a quantum measuring device can be implemented by

(1) applying an invertible dynamics (the isometry V) that transforms the input system $\mathcal{K}_2$ into the composite system $\mathcal{K}_1 \otimes \mathcal{V}$, where $\mathcal{V}$ is an ancillary system;
(2) performing the given measurement $\mathcal{E}$ on $\mathcal{K}_1$, thus obtaining the outcome $i \in X$;
(3) conditionally to the outcome $i \in X$, performing a quantum measurement (the predual instrument $\mathcal{J}_{i\,*}$), thus obtaining an outcome in Ω and transforming the ancillary system $\mathcal{V}$ into the classical system described by the commutative algebra $\ell^\infty(Y)$.

When Ω is a countable set, we have that the instrument $\mathcal{J} : \sigma(\Omega) \to \mathrm{CP}\,(\ell^\infty(Y), \ell^\infty(X, \mathcal{L}(\mathcal{V})))$ is completely specified by its action on singleton sets, that is, by the quantum operations $\{\mathcal{J}_\omega \in \mathrm{CP}\,(\ell^\infty(Y), \ell^\infty(X, \mathcal{L}(\mathcal{V}))) \mid \omega \in \Omega\}$. In this case, if for all $i \in X$ we set

$$Q^{(i)}_{\omega,j} := \mathcal{J}_\omega(\delta_j)_i = \mathcal{J}_{i,\omega}(\delta_j) \quad \forall(\omega, j) \in \Omega \times Y\,,$$

then the map $(\omega, j) \mapsto Q^{(i)}_{\omega,j}$ is a normalized POVM on the product set $\Omega \times Y$ and with values in $\mathcal{L}(\mathcal{V})$. Note that, in terms of the POVM $Q^{(i)}$, we can express each $\mathcal{J}_{i,\omega}$ as

$$\mathcal{J}_{i,\omega}(f) = \sum_{j\in Y} f_j\, Q^{(i)}_{\omega,j} \quad \forall f \in \ell^\infty(Y)$$

or, equivalently,

$$(\mathcal{J}_{i,\omega\,*}(\sigma))_j = \mathrm{tr}\left(\sigma Q^{(i)}_{\omega,j}\right) \quad \forall \sigma \in \mathcal{T}\,(\mathcal{V})\;.$$

In other words, the step (3) in the measurement process can be interpreted as a quantum measurement with outcome $(\omega, j) \in \Omega \times Y$, where only the classical information concerning the index $j \in Y$ is encoded in a physical system available for future experiments, whereas the information concerning index $\omega \in \Omega$ becomes unavailable after being red out by the experimenter.

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