Covariant positive operator valued measures and instruments: an overview

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Mathematical setting

- $\mathcal{H}$ = Hilbert space of the quantum system
- $\mathcal{L}(\mathcal{H})$ = bounded operators on $\mathcal{H}$
- $\mathcal{U}(\mathcal{H})$ = unitary operators on $\mathcal{H}$
- $\mathcal{T}(\mathcal{H})$ = trace class operators
- $\mathcal{S}(\mathcal{H}) = \{S \in \mathcal{T}(\mathcal{H}) \mid S \geq 0, \operatorname{tr}[S] = 1\}$ = state space
- $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ = bounded operators in $\mathcal{T}(\mathcal{H})$
- $(\Omega, \mathcal{A})$ = measurable outcome space

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Observables

Definition

An *observable* (or *positive operator valued measure*, or *POVM*) is a mapping $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that

1. $\mathsf{E}(X) \geq 0$ for all $X \in \mathcal{A}$
2. $\mathsf{E}(\Omega) = 1$
3. $\mathrm{WOT}-\sum_i \mathsf{E}(X_i) = \mathsf{E}(\cup_i X_i)$ if $\{X_i\}_{i\in\mathbb{N}}$ is a sequence such that $X_i \cap X_j = \emptyset$ for $i \neq j$.

The probability measure p_S^{E} on Ω

$$p_S^{\mathsf{E}}(X) = \mathrm{tr}\left[S\mathsf{E}(X)\right]$$

describes the statistics of a measurement of E performed on S.

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An *instrument* (or *completely positive (CP) operator valued measure*) is a mapping $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ such that

1. $\mathcal{I}(X)$ is CP for all $X \in \mathcal{A}$
2. $\mathrm{tr}\,[[\mathcal{I}(\Omega)](T)] = \mathrm{tr}\,[T]$ for all $T \in \mathcal{T}(\mathcal{H})$
3. $\mathrm{strong}-\sum_i \mathcal{I}(X_i) = \mathcal{I}(\cup_i X_i)$ if $\{X_i\}_{i\in\mathbb{N}}$ is a sequence such that $X_i \cap X_j = \emptyset$ for $i \neq j$.

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Notation

$$\mathcal{I}_X := \mathcal{I}(X) \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$$

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The *associated observable* $\mathsf{E}^{\mathcal{I}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is

$$\mathrm{tr}\left[\mathsf{E}^{\mathcal{I}}(X)T\right] := \mathrm{tr}\,[[\mathcal{I}(X)](T)] \qquad \forall T \in \mathcal{T}(\mathcal{H})$$

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If $S \in \mathcal{S}(\mathcal{H})$, its *a posteriori state* is

$$S_X = \mathcal{I}_X(S)/\mathrm{tr}\,[\mathcal{I}_X(S)] \qquad (0/0 = 0)$$

Group actions

A *simmetry group* G acts

- in $\mathcal{S}(\mathcal{H})$ by means of a projective unitary representation U as
$$g[S] = U(g)SU(g)^{-1} \qquad g \in G,\ S \in \mathcal{S}(\mathcal{H})$$
- in Ω by means of a measurable action
$$g[\omega] \qquad g \in G,\ \omega \in \Omega$$

Technical assumptions:

- Ω is a locally compact second countable (lcsc) space, with $\mathcal{A} = \mathcal{B}(\Omega)$
- G is a lcsc topological group
- the map $G \times \Omega \ni (g, \omega) \mapsto g[\omega] \in \Omega$ is continuous

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Projective representation

$U : G \to \mathcal{U}(\mathcal{H})$ is a *projective representation* if

$$U(g)U(h) = m(g,h)U(gh)$$

with $|m(g,h)| = 1$ and $m(g,g^{-1}) = 1$

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Covariant measurements

Definition

An observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is *covariant* if

$$U(g)\mathsf{E}(X)U(g)^{-1} = \mathsf{E}(g[X])$$

for all $X \in \mathcal{A}$, $g \in G$

Definition

An instrument $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ is *covariant* if

$$U(g)\mathcal{I}_X(U(g)^{-1}TU(g))U(g)^{-1} = \mathcal{I}_{g[X]}(T)$$

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Remark

This means

$$p^{\mathsf{E}}_{g[S]}(g[X]) = p^{\mathsf{E}}_{S}(X)$$

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$$(g[S])_{g[X]} = g[S_X] \qquad \forall S \in \mathcal{S}(\mathcal{H})$$

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for all $X \in \mathcal{A}$, $g \in G$, $T \in \mathcal{T}(\mathcal{H})$

Example: position and momentum (in dim. 1)

- $\mathcal{H} = L^2(\mathbb{R})$
- $\Omega = \begin{cases} \mathbb{R} & = & \text{position} \\ \mathbb{P} & = & \text{momentum} \\ \mathbb{R} \times \mathbb{P} & = & \text{phase-space} \end{cases}$
- $G = \left\{ \begin{array}{ccc} T & = & \text{group of translation} \\ B & = & \text{group of boosts} \end{array} \right\} \simeq \mathbb{R}$
- $$U : T \to \mathcal{U}(\mathcal{H}) \qquad [U(a)f](x) = f(x-a)$$
  $$V : B \to \mathcal{U}(\mathcal{H}) \qquad [V(p)f](x) = e^{ipx} f(x)$$

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$\mathsf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is a *position observable* if

$$U(a)\mathsf{E}(X)U(a)^{-1} = \mathsf{E}(X + a) \qquad V(p)\mathsf{E}(X)V(p)^{-1} = \mathsf{E}(X)$$

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Example

The *canonical position observable* is

$$[\mathsf{Q}(X)f] = 1_X(x)f(x) \qquad \forall f \in L^2(\mathbb{R})$$

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$$U(a)\mathsf{E}(X)U(a)^{-1} = \mathsf{E}(X + a) \qquad V(p)\mathsf{E}(X)V(p)^{-1} = \mathsf{E}(X)$$

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Example: position and momentum (in dim. 1)

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Example

The *canonical momentum observable* is

$$\mathsf{P}(Y) = \mathcal{F}^{-1}\mathsf{Q}(Y)\mathcal{F}$$

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Definition

$\mathsf{G} : \mathcal{B}(\mathbb{R} \times \mathbb{P}) \to \mathcal{L}(\mathcal{H})$ is a *phase-space observable* if

$$\begin{aligned} U(a)\mathsf{G}(X \times Y)U(a)^{-1} &= \mathsf{F}((X + a) \times Y) \\ V(p)\mathsf{F}(X \times Y)V(p)^{-1} &= \mathsf{F}(X \times (Y + p)) \end{aligned}$$

Example: position and momentum (in dim. 1)

In terms of a single irreducible projective representation

$$W : T \times B \to \mathcal{U}(\mathcal{H}) \qquad W(a,p) = U(a)V(p)$$

and actions

$$\begin{aligned} (a,p)[x]_1 &= x + a && \text{on } \mathbb{R} \\ (a,p)[y]_2 &= y + p && \text{on } \mathbb{P} \\ (a,p)[(x,y)] &= (x + a, y + p) && \text{on } \mathbb{R} \times \mathbb{P} \end{aligned}$$

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Remark

The actions change, but the group representation is the same

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Example: phase observable for the photon

- $\mathcal{H} = \mathrm{span}\,\{|n\rangle \mid n = 0, 1, 2 \ldots\}$ = state space for the single mode optical field
- $|z\rangle = e^{-|z|^2/2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle$ = monochromatic laser light of energy $|z|$ and phase $\arg z$
- $(\Omega, \mathcal{A}) = ([0, 2\pi], \mathcal{B}([0, 2\pi]))$ = phase

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For a phase observable E we require

$$p^{\mathsf{E}}_{|ze^{i\theta}\rangle}(X) = p^{\mathsf{E}}_{|z\rangle}(X + \theta \pmod{2\pi}) \qquad \forall X \in \mathcal{B}(\Omega)$$

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Number operator

$N = a^* a$, with a, a^* the lowering and raising operators

Defi

E : $\mathcal{B}$(

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Indice

Systems of imprimitivity

Definition

A *system of imprimitivity* based on Ω is a triple $(V, \mathrm{P}, \mathcal{K})$, where

- V is a unitary representation of G in the Hilbert space $\mathcal{K}$
- $\mathrm{P} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{K})$ is a projection valued measure
- for all $X \in \mathcal{B}(\Omega)$, $g \in G$

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PVMs

A *projection valued measure* is a POVM P satisfying

$$\mathsf{P}(X)\mathsf{P}(Y) = \mathsf{P}(X \cap Y)$$

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Dilation theorem for observables

Theorem (Cattaneo, 1979)

E *is an U-covariant POVM based on* Ω *iff there exists*

- *a system of imprimitivity* $(V, \mathsf{P}, \mathcal{K})$ *based on* Ω
- *an isometry* $W : \mathcal{H} \to \mathcal{K}$ *satisfying*

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$\mathcal{I}$ is an U-covariant instrument based on Ω iff there exists

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such that, for all $X \in \mathcal{B}(\Omega)$ and $T \in \mathcal{T}(\mathcal{H})$,

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Inducing functor and imprimitivity theorem

Basic assumptions

In the following

- H is a closed subgroup of G
- Ω is the homogeneous space G/H

Induction is a functor

$$\mathrm{Rep}\,(H) \ni \sigma \longmapsto (V^{\sigma}, \mathrm{P}^{\sigma}, \mathcal{K}^{\sigma}) \in \mathrm{Impr}\,(G, G/H)$$

Theorem (Mackey)

Induction is an equivalence of categories.

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Example

If $H = \{1\}$ and $\sigma = 1$, then

- $\mathcal{K}^\sigma = L^2(G, \mu_G)$
- $[V^\sigma(g)f](h) = f(g^{-1}h)$
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General problem

For all $\sigma \in \mathrm{Rep}(H)$

Diagonalize $(V^\sigma, \mathcal{K}^\sigma)$

$\Downarrow$

Find all possible isometries $W : \mathcal{H} \to \mathcal{K}^\sigma$ such that $WU(g) = V^\sigma(g)W \; \forall g \in G$

$\Downarrow$

Write down explicitely $W^* \mathsf{P}^\sigma(X) W$ and $\mathrm{tr}_{\mathcal{K}} [(\mathsf{P}(X) \otimes I_{\mathcal{H}}) WTW^*]$

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Particular solutions

A complete solution is available in the following cases

(1) G generic, H compact, U irreducible
(2) G abelian, H generic, U generic
(3) G compact, H generic, U generic
(4) $G = H \ltimes K$, H normal and abelian, U irreducible

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Phase-space observables and instruments $\in$ (1)

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Example

Position and momentum observables $\in$ (4)

G generic, H compact, U irreducible

Theorem (Cassinelli, De Vito, Toigo, 2003)

Suppose U projective and irreducible, $H \subset G$ compact. Then there exists U-covariant POVMs on G/H iff U is square integrable.

In this case, there exists a selfadjoint positive operator $C : \mathcal{H} \to \mathcal{H}$ such that every U-covariant POVM on G/H is of the form

$$\mathsf{E}(X) = \int_X U(g) CSCU(g)^{-1} \, d\mu_{G/H}(gH) \qquad \forall X \in \mathcal{B}(G/H)$$

with $S \in \mathcal{S}(\mathcal{H})$ such that

$$SU(h) = U(h)S \qquad \forall h \in H.$$

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Square integrable representations

U is *square integrable* if there exists a nonzero $v \in \mathcal{H}$ such that

$$\int_G |\langle v , U(g)v \rangle|^2 \, \mathrm{d}\mu_G(g) < \infty$$

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Formal degree

- The selfadjoint operator C^2 is the *formal degree of G*
- $C = \lambda I$ if G is unimodular, and is unbounded if G is non-unimodular
- $U(g)C = \Delta_G(g)^{-1/2} CU(g)$ for all $g \in G$

G generic, H compact, U irreducible

Theorem (Carmeli, Heinosaari, Toigo 2009)

Suppose U projective irreducible, H compact. Then there exists U-covariant instruments on G/H iff U is square integrable.

In this case, if C is as before, every U-covariant instrument on G/H is of the form

$$\mathcal{I}^*_X(A) = \int_X U(g) C \Phi(U(g)^* A U(g)) C U(g)^{-1} \, d\mu_{G/H}(gH)$$

for all $X \in \mathcal{B}(G/H)$, $A \in \mathcal{L}(\mathcal{H})$, where $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is such that

- *Φ is normal and CP*
- $\mathrm{tr}\,[\Phi(I)] = 1$
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Theorem (Carmeli, Heinosaari, Toigo 2009)

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Position and momentum observables

Theorem

E *is a position observable iff there exists a probability density* ρ *on* $\mathbb{R}$ *such that*

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Remark

It follows that the associated probability distributions are just convolutions with the canonical ones

$$p_S^{\mathsf{E}_\rho} = \rho * p_S^{\mathsf{Q}}$$
$$p_S^{\mathsf{F}_\nu} = \nu * p_S^{\mathsf{P}}$$

Phase space

- There exists phase space observables, and every phase space observable is of the form

$$\mathsf{G}(X \times Y) = \frac{1}{2\pi} \iint_{X \times Y} U(a)V(p)SV(p)^* U(a)^* \, \mathrm{d}a \, \mathrm{d}p$$

with $S \in \mathcal{S}(\mathcal{H})$

- The margins of G

$$\begin{aligned} \mathsf{E}(X) &:= \mathsf{G}(X \times \mathbb{P}) \\ \mathsf{F}(Y) &:= \mathsf{G}(\mathbb{R} \times Y) \end{aligned}$$

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Position and momentum observables E_ρ, F_ν are *coexistent* if they are margins of a single observable G on $\mathbb{R} \times \mathbb{P}$

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- If E_ρ and F_ν are coexistent, then ρ, ν are absolutely continuous wrt the Lebesgue measure and Fourier-related
- $\mathrm{Var}\left(p_S^{\mathsf{E}_\rho}\right) \mathrm{Var}\left(p_S^{\mathsf{F}_\nu}\right) \geq 1$ for all $S \in \mathcal{S}(\mathcal{H})$
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Position and momentum instruments

Definition

An instrument $\mathcal{I} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ is a *position instrument* if

$$\begin{aligned} U(a)\mathcal{I}_X(U(a)^* T U(a))U(a)^* &= \mathcal{I}_{X+a}(T) \\ V(p)\mathcal{I}_X(V(p)^* T V(p))V(p)^* &= \mathcal{I}_X(T) \end{aligned}$$

A similar definition holds for *momentum instruments*. In particular, $\mathcal{J} : \mathcal{B}(\mathbb{P}) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$ is a momentum instrument iff

$$\mathcal{I}_X(T) = \mathcal{F}\mathcal{J}_X(\mathcal{F}^{-1} T \mathcal{F})\mathcal{F}^{-1}$$

defines a position instrument $\mathcal{I}$.

Structure theorem for position instruments

Theorem

There is a one-to-one correspondence between position instruments and couples (μ, K), with

- *μ probability measure on $\mathbb{R}$*
- *$K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ positive semidefinite kernel satisfying*

$$\int K(x, h; x, h)\, dx\, d\mu(h) = 1$$

If (μ, K) is as above, the corresponding position instrument is

$$[\mathcal{I}_X(T)](x, y) = \int 1_X(z) T(x+h, y+h) K(z-y, h; z-x, h)\, dz\, d\mu(h)$$

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Positive definite kernels

A map $K : X \times X \to \mathbb{C}$ is $\text{positive semidefinite}$ if for all $N \in \mathbb{Z}_+$, $\{x_i\}_{i=1,2\ldots N} \subset X$, $\{c_i\}_{i=1,2\ldots N} \subset \mathbb{C}$

$$\sum_{i,j=1}^{N} c_i \overline{c_j} K(x_i, x_j) \geq 0$$

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Remark

Trace class operators have been identified with their associated kernels, i. e.

$$Tf(x) = \int T(x,y)f(y)\,\mathrm{d}y \qquad \forall f \in L^2(\mathbb{R})$$

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G abelian, H generic, U generic

- Dual group:

$$\hat{G} = \{\gamma \in C(G) \mid \gamma(gh) = \gamma(g)\gamma(h)\ \forall g, h \in G\}$$

- Annihilator of H in $\hat{G}$:

$$H^{\perp} = \left\{\gamma \in \hat{G} \mid \gamma(h) = 1\ \forall h \in H\right\}$$

- Quotient map:

$$p : \hat{G} \to \hat{G}/H^{\perp}$$

We have identifications

$$\hat{H} = \hat{G}/H^{\perp} \qquad \text{(canonical)}$$

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There exists U-covariant POVMs on G/H iff $\nu = \alpha\, p(\nu) \otimes \mu_{H^\perp}$.
In this case, every U-covariant POVM on G/H has the form

$$\langle E(X)\psi\,,\,\phi\rangle = \int_{\hat{G}} d\nu(\gamma) \int_{H^\perp} d\mu_{H^\perp}(\chi) \int_X d\mu_{G/H}(gH)$$
$$\sqrt{\alpha(\gamma)\alpha(\gamma\chi^{-1})}\, \chi(gH) \left\langle W(\gamma\chi^{-1})\psi(\gamma\chi^{-1})\,,\, W(\gamma)\phi(\gamma) \right\rangle$$

for some choice of a map

$$\hat{G} \ni \gamma \mapsto W_\gamma \in \mathcal{L}(\mathcal{K}; \mathcal{K}_\infty) \qquad W^*_\gamma W_\gamma = I_{\mathcal{K}}$$

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The action of U in $L^2\left(\hat{G}, \nu; \mathcal{K}\right)$ *is*

$$[U(g)\phi](\gamma) = \gamma(g)\phi(\gamma)$$

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$$\bigoplus_{n \in \mathbb{Z}_+ \cup \{\infty\}} U_n$$

with U_n *acting in* $L^2\left(\hat{G}, \nu_n; \mathcal{K}_n\right)$ *as above,*
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$$\mathcal{H} = \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathcal{K}_\pi$$

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where $\hat{G}$ is the set of irreducible unitary representations $(\pi, \mathcal{H}_\pi)$ of G

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3. $\mathrm{tr}_{\mathcal{H}_\rho}(K(\pi, \pi)) = \delta_{\rho\pi} d_\rho I_{\mathcal{K}_\rho}$

G compact, H compact, U generic

$$\mathcal{H} = \bigoplus_{\pi \in \hat{G}} \mathcal{H}_\pi \otimes \mathcal{K}_\pi$$

$$U = \bigoplus_{\pi \in \hat{G}} \pi \otimes I_{\mathcal{K}_\pi}$$

Let $\mathcal{C}$ be the convex set of maps $K : \hat{G} \times \hat{G} \to \mathcal{L}(\mathcal{H})$ s. t.

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Theorem

There is a one-to-one convex correspondence betwteen $\mathcal{C}$ and the set of U-covariant POVMs on G/H, given by

$$\langle E(X)v_\pi \, , \, w_\rho \rangle = \int_X \langle K(\rho,\pi)U(g)^* v_\pi \, , \, U(g)^* w_\rho \rangle \, d\mu_{G/H}(gH)$$

for all $v_\pi \in \mathcal{H}_\pi \otimes \mathcal{K}_\pi$, $w_\rho \in \mathcal{H}_\rho \otimes \mathcal{K}_\rho$

An application: extremal POVMs

- The POVMs on a set Ω form a convex set $\mathcal{P}(\Omega)$, which is compact in a suitable weak-* topology
- If $\mathsf{E} \notin \operatorname{ext} \mathcal{P}(\Omega)$, then $\mathsf{E} = p_1 \mathsf{E}_1 + p_2 \mathsf{E}_2$ with $p_1 + p_2 = 1$, i. e. E comes from the random choice between two different observables
- Every $\mathsf{E} \in \mathcal{P}(\Omega)$ can be approximated by the convex sum of elements in $\operatorname{ext} \mathcal{P}(\Omega)$
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For $K \in \mathcal{C}$, let $\mathcal{H}_K$ be the RKHS associated to K.

$\mathcal{H}_K$ carries an unitary representation $\tilde{U}$ of H, given by

$$[\tilde{U}(h)f](\pi) = U(h)f(\pi)$$

$$\mathcal{T}_U := \{T \in \mathcal{T}(\mathcal{H}) \mid TU(g) = U(g)T \ \forall g \in G\}$$

$$\mathcal{T}_{\tilde{U}} := \left\{T \in \mathcal{T}(\mathcal{H}_K) \mid T\tilde{U}(h) = \tilde{U}(h)T \ \forall h \in H\right\}$$

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For $K \in \mathcal{C}$, let $\mathcal{H}_K$ be the RKHS associated to K.

$\mathcal{H}_K$ (

Reproducing kernel Hilbert space associated to K

- $\mathcal{H}_K^0 = \mathrm{span}\left\{ K(\cdot, \pi)v \mid \pi \in \hat{G},\ v \in \mathcal{H} \right\}$
- $\langle \cdot, \cdot \rangle_K : \mathcal{H}_K^0 \times \mathcal{H}_K^0 \to \mathbb{C}$ given by

$$\langle K(\cdot,\pi)v , K(\cdot,\rho)w \rangle_K = \langle K(\rho,\pi)v , w \rangle$$

Then the completion $\mathcal{H}_K$ of $\mathcal{H}_K^0$ is a Hilbert space of $\mathcal{H}$-valued functions in which the evaluation maps ev_π are continuous

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Theorem

The POVM associated to K is extremal in the set of U-covariant POVMs on G/H iff $\tilde{\mathcal{T}}_U = \mathcal{T}_{\tilde{U}}$.

Example: extremal phase observables

Let E be a phase observable with associated matrix $\{c_{mn}\}_{m,n\in\mathbb{N}}$. Let

- $\{\eta_n\}_{n\in\mathbb{N}}$ be vectors in $\mathcal{H}$ such that $c_{mn} = \langle \eta_m , \eta_n \rangle$
- $\mathcal{H}_0$ be their closed linear span.

Then E is extremal in the convex set of phase observables iff

$$\mathcal{T}(\mathcal{H}_0) = \overline{\mathrm{span}} \, \{ |\eta_n\rangle \langle \eta_n| \mid n \in \mathbb{N} \}$$

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