# Covariant positive operator valued measures and instruments: an overview

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- Symmetry groups and covariant measurements
- Examples

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- General structure theorems
- Explicit structure theorems and examples

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# Indice

### Measurements in Quantum Mechanics

- Observables and instruments
- Symmetry groups and covariant measurements
- Examples

#### Structure theorems

- General structure theorems
- Explicit structure theorems and examples

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# Mathematical setting

- $\mathcal{H} = \text{Hilbert space of the quantum system}$
- $\mathcal{L}(\mathcal{H}) =$  bounded operators on  $\mathcal{H}$
- $\mathcal{U}(\mathcal{H}) =$  unitary operators on  $\mathcal{H}$
- T(H) = trace class operators
- $\mathcal{S}(\mathcal{H}) = \{ \mathcal{S} \in \mathcal{T}(\mathcal{H}) \mid \mathcal{S} \ge 0, \, \mathrm{tr}\, [\mathcal{S}] = 1 \} = \mathrm{state space}$
- $\mathcal{L}(\mathcal{T}(\mathcal{H})) =$ bounded operators in  $\mathcal{T}(\mathcal{H})$
- $(\Omega, \mathcal{A})$  = measurable outcome space

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# Observables

#### Definition

An observable (or positive operator valued measure, or POVM) is a mapping  $E:\mathcal{A}\to\mathcal{L}(\mathcal{H})$  such that

• 
$$\mathsf{E}(X) \ge 0$$
 for all  $X \in \mathcal{A}$ 

**2** 
$$\mathsf{E}(\Omega) = 1$$

③ WOT− $\sum_{i} E(X_i) = E(\cup_i X_i)$  if  $\{X_i\}_{i \in \mathbb{N}}$  is a sequence such that  $X_i \cap X_j = \emptyset$  for  $i \neq j$ .

The probability measure  $p_S^{\mathsf{E}}$  on  $\Omega$ 

$$\mathcal{D}_{S}^{\mathsf{E}}(X) = \operatorname{tr}\left[S\mathsf{E}(X)\right]$$

describes the statistics of a measurement of E performed on S.

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### Instruments

#### Definition

An *instrument* (or *completely positive* (*CP*) *operator valued measure*) is a mapping  $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$  such that

- $\mathcal{I}(X)$  is CP for all  $X \in \mathcal{A}$
- 2 tr  $[[\mathcal{I}(\Omega)](\mathcal{T})] = tr [\mathcal{T}]$  for all  $\mathcal{T} \in \mathcal{T}(\mathcal{H})$
- Strong− $\sum_{i} \mathcal{I}(X_i) = \mathcal{I}(\cup_i X_i)$  if  $\{X_i\}_{i \in \mathbb{N}}$  is a sequence such that  $X_i \cap X_j = \emptyset$  for  $i \neq j$ .

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Notation

$$\mathcal{I}_X := \mathcal{I}(X) \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$$

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The associated observable 
$$\mathsf{E}^{\mathcal{I}} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$$
 is  
 $\operatorname{tr} \left[\mathsf{E}^{\mathcal{I}}(X)\mathcal{T}\right] := \operatorname{tr} \left[\left[\mathcal{I}(X)\right](\mathcal{T})\right] \quad \forall \mathcal{T} \in \mathcal{T}(\mathcal{H})$ 

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If 
$$S \in \mathcal{S}(\mathcal{H})$$
, its a posteriori state is $S_X = \mathcal{I}_X(S)/\mathrm{tr}\left[\mathcal{I}_X(S)
ight] \qquad (0/0=0)$ 

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# Group actions

### A simmetry group G acts

- in S(H) by means of a projective unitary representation U
   as
   g[S] = U(g)SU(g)<sup>-1</sup>
   g ∈ G, S ∈ S(H)
- in  $\Omega$  by means of a measurable action

 $g[\omega] \qquad g \in G, \, \omega \in \Omega$ 

Technical assumptions:

- $\Omega$  is a locally compact second countable (lcsc) space, with  $\mathcal{A} = \mathcal{B}(\Omega)$
- G is a lcsc topological group
- the map  $G imes \Omega 
  i (g, \omega) \mapsto g[\omega] \in \Omega$  is continuous

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U(q)U(h) = m(q, h)U(qh)

• in  $\Omega$  by means of a measurable action

#### Projective representation

 $U: G \rightarrow \mathcal{U}(\mathcal{H})$  is a projective representation if

with |m(g, h)| = 1 and  $m(g, g^{-1}) = 1$ 

 $\mathcal{A} = \mathcal{D}(\mathcal{M})$ 

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### Covariant measurements

#### Definition

An observable  $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$  is *covariant* if  $U(g)\mathsf{E}(X)U(g)^{-1} = \mathsf{E}(g[X])$ 

for all  $X \in \mathcal{A}, g \in G$ 

#### Definition

An instrument  $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$  is *covariant* if  $U(g)\mathcal{I}_X(U(g)^{-1}TU(g))U(g)^{-1} = \mathcal{I}_{g[X]}(T)$ for all  $X \in \mathcal{A}, g \in G, T \in \mathcal{T}(\mathcal{H})$ 

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#### Remark Defir This means

$$p^{\mathsf{E}}_{g[S]}(g[X]) = p^{\mathsf{E}}_{S}(X)$$

 $U(g)\mathcal{I}_X(U(g)^{-1}TU(g))U(g)^{-1}=\mathcal{I}_{g[X]}(T)$  for all  $X\in\mathcal{A},\,g\in G,\,T\in\mathcal{T}(\mathcal{H})$ 

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#### Definition

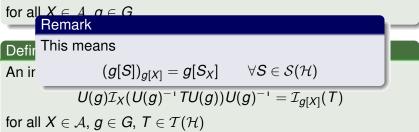
An instrument  $\mathcal{I} : \mathcal{A} \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$  is *covariant* if  $U(g)\mathcal{I}_X(U(g)^{-1}TU(g))U(g)^{-1} = \mathcal{I}_{g[X]}(T)$ for all  $X \in \mathcal{A}, g \in G, T \in \mathcal{T}(\mathcal{H})$ 

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# Example: position and momentum (in dim. 1)

• 
$$\mathcal{H} = L^2(\mathbb{R})$$
  
•  $\Omega = \begin{cases} \mathbb{R} = \text{position} \\ \mathbb{P} = \text{momentum} \\ \mathbb{R} \times \mathbb{P} = \text{phase-space} \end{cases}$   
•  $G = \begin{cases} T = \text{group of translation} \\ B = \text{group of boosts} \end{cases} \simeq \mathbb{R}$   
 $U: T \to \mathcal{U}(\mathcal{H}) \qquad [U(a)f](x) = f(x-a)$   
•  $V: B \to \mathcal{U}(\mathcal{H}) \qquad [V(p)f](x) = e^{ipx}f(x)$ 

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# Example: position and momentum (in dim. 1)

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$$\mathcal{H} = L^2(\mathbb{R})$$
  
•  $\Omega = \begin{cases} \mathbb{R} = \text{position} \\ \mathbb{P} = \text{momentum} \\ \mathbb{R} \times \mathbb{P} = \text{phase-space} \end{cases}$   
•  $G = \begin{cases} T = \text{group of translation} \\ B = \text{group of boosts} \end{cases} \simeq \mathbb{R}$   
 $U: T \to \mathcal{U}(\mathcal{H}) \qquad [U(a)f](x) = f(x-a)$   
•  $V: B \to \mathcal{U}(\mathcal{H}) \qquad [V(p)f](x) = e^{ipx}f(x)$ 

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Measurements in Quantum Mechanics Structure theorems References Examples Example: position and momentum (in dim. 1) •  $\mathcal{H} = L^2(\mathbb{R})$ •  $\Omega = \begin{cases} \mathbb{R} &= \text{position} \\ \mathbb{P} &= \text{momentum} \\ \mathbb{R} \times \mathbb{P} &= \text{phase-space} \end{cases}$ •  $G = \left\{ \begin{array}{ll} T = \text{group of translation} \\ B = \text{group of boosts} \end{array} \right\} \simeq \mathbb{R}$  $U: T \to \mathcal{U}(\mathcal{H})$  [U(a)f](x) = f(x-a) $V: B \to \mathcal{U}(\mathcal{H})$   $[V(p)f](x) = e^{ipx}f(x)$ 

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## Example: position and momentum (in dim. 1)

#### Definition

 $\mathsf{E}:\mathcal{B}(\mathbb{R})\to\mathcal{L}(\mathcal{H})$  is a position observable if

 $U(a)E(X)U(a)^{-1} = E(X + a)$   $V(p)E(X)V(p)^{-1} = E(X)$ 

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### Example

The canonical position observable is

$$[\mathsf{Q}(X)f] = \mathsf{1}_X(x)f(x) \qquad \forall f \in L^2(\mathbb{R})$$

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#### Definition

 $F : \mathcal{B}(\mathbb{P}) \to \mathcal{L}(\mathcal{H})$  is a momentum observable if  $U(a)F(Y)U(a)^{-1} = F(Y) \qquad V(p)F(Y)V(p)^{-1} = F(Y+p)$ 

# Example: position and momentum (in dim. 1)

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#### Definition

$$\begin{split} \mathsf{F} &: \mathcal{B}(\mathbb{P}) \to \mathcal{L}(\mathcal{H}) \text{ is a momentum observable if} \\ & U(a)\mathsf{F}(Y)U(a)^{-1} = \mathsf{F}(Y) \qquad V(p)\mathsf{F}(Y)V(p)^{-1} = \mathsf{F}(Y+p) \end{split}$$

### Example

The canonical momentum observable is

$$\mathsf{P}(Y) = \mathcal{F}^{-1}\mathsf{Q}(Y)\mathcal{F}$$

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#### Definition

 $G:\mathcal{B}(\mathbb{R}\times\mathbb{P})\to\mathcal{L}(\mathcal{H})$  is a phase-space observable if

$$U(a)G(X \times Y)U(a)^{-1} = F((X + a) \times Y)$$
  
$$V(p)F(X \times Y)V(p)^{-1} = F(X \times (Y + p))$$

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## Example: position and momentum (in dim. 1)

In terms of a single irreducible projective representation

 $W: T \times B \rightarrow \mathcal{U}(\mathcal{H})$  W(a, p) = U(a)V(p)

and actions

$$(a,p)[x]_1 = x + a \quad \text{on } \mathbb{R}$$
  

$$(a,p)[y]_2 = y + p \quad \text{on } \mathbb{P}$$
  

$$(a,p)[(x,y)] = (x + a, y + p) \quad \text{on } \mathbb{R} \times \mathbb{P}$$

we have

$$W(x,p)E(X)W(x,p)^{-1} = E((x,p)[X]_1) W(x,p)F(Y)W(x,p)^{-1} = F((x,p)[Y]_2) W(x,p)G(Z)W(x,p)^{-1} = E((x,p)[Z])$$

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Example: position and momentum (in dim. 1)

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 $W: T \times B \rightarrow \mathcal{U}(\mathcal{H})$  W(a, p) = U(a)V(p)

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$$\begin{array}{rcl} (a,p)[x]_1 &=& x+a & \mathrm{on} \ \mathbb{R} \\ (a,p)[y]_2 &=& y+p & \mathrm{on} \ \mathbb{P} \\ (a,p)[(x,y)] &=& (x+a,y+p) & \mathrm{on} \ \mathbb{R} \times \mathbb{P} \end{array}$$

we have

$$W(x,p)E(X)W(x,p)^{-1} = E((x,p)[X]_1) W(x,p)F(Y)W(x,p)^{-1} = F((x,p)[Y]_2) W(x,p)G(Z)W(x,p)^{-1} = E((x,p)[Z])$$

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Example: position and momentum (in dim. 1)

In terms of a single irreducible projective representation

 $W: T \times B \rightarrow \mathcal{U}(\mathcal{H})$  W(a, p) = U(a)V(p)

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 $(a,p)[x]_{1} = x + a \quad \text{on } \mathbb{R}$   $(a,p)[y]_{2} = y + p \quad \text{on } \mathbb{P}$ Remark
The actions change, but the group representation is the same  $W(x,p) E(X) W(x,p)^{-1} = E((x,p)[X]_{1})$   $W(x,p) F(Y) W(x,p)^{-1} = F((x,p)[Y]_{2})$   $W(x,p) G(Z) W(x,p)^{-1} = E((x,p)[Z])$ 

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- *H* = span { |*n*⟩ | *n* = 0, 1, 2...} = state space for the single mode optical field
- $|z\rangle = e^{-|z|^2/2} \sum_{n \frac{z^n}{\sqrt{n!}}} |n\rangle =$  monochromatic laser light of energy |z| and phase arg z

• 
$$(\Omega, \mathcal{A}) = ([0, 2\pi], \mathcal{B}([0, 2\pi])) =$$
phase

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### Example: phase observable for the photon

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For a phase observable E we require

$$p^{\mathsf{E}}_{|{m z} e^{i heta}
angle}(X) = p^{\mathsf{E}}_{|{m z}
angle}(X + heta\,(\mathrm{mod}\;2\pi)) \qquad orall X \in \mathcal{B}(\Omega)$$

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$$(\Omega, \mathcal{A}) = ([0, 2\pi], \mathcal{B}([0, 2\pi])) =$$
phase

### Definition

$$\begin{split} \mathsf{E} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H}) \text{ is a } phase \ observable \ \mathsf{if} \\ e^{i\theta N} \mathsf{E}(X) e^{-i\theta N} = \mathsf{E}(X + \theta \pmod{2\pi}) ) \\ \mathsf{for \ all} \ X \in \mathcal{B}(\Omega), \ \theta \in [0, 2\pi] \end{split}$$

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#### • $(\Omega, A) = ([0, 2\pi], \mathcal{B}([0, 2\pi])) = \text{phase}$ Number operator

Define  $N = a^*a$ , with  $a, a^*$  the lowering and raising **E** :  $\mathcal{B}$  operators

$$e^{i\theta N}\mathsf{E}(X)e^{-i\theta N}=\mathsf{E}(X+\theta \pmod{2\pi})$$

for all  $X \in \mathcal{B}(\Omega)$ ,  $\theta \in [0, 2\pi]$ 

General structure theorems Explicit structure theorems and examples

## Indice

### Measurements in Quantum Mechanics

- Observables and instruments
- Symmetry groups and covariant measurements
- Examples

### 2 Structure theorems

- General structure theorems
- Explicit structure theorems and examples

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# Systems of imprimitivity

### Definition

A system of imprimitivity based on  $\Omega$  is a triple (V, P,  $\mathcal{K}$ ), where

- V is a unitary representation of G in the Hilbert space  $\mathcal{K}$
- $P : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{K})$  is a projection valued measure
- for all  $X \in \mathcal{B}(\Omega)$ ,  $g \in G$

 $V(g)\mathsf{P}(X)V(g)^* = \mathsf{P}(g[X])$ 

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General structure theorems Explicit structure theorems and examples

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### PVMs

A projection valued measure is a POVM P satisfying

$$\mathsf{P}(X)\mathsf{P}(Y)=\mathsf{P}(X\cap Y)$$

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## Dilation theorem for observables

#### Theorem (Cattaneo, 1979)

- E is an U-covariant POVM based on  $\Omega$  iff there exists
  - a system of imprimitivity (V, P,  $\mathcal{K}$ ) based on  $\Omega$
  - an isometry  $W : \mathcal{H} \to \mathcal{K}$  satisfying

 $WU(g) = V(g)W \quad \forall g \in G$ 

such that, for all  $X \in \mathcal{B}(\Omega)$ ,

 $\mathsf{E}(X) = W^* \mathsf{P}(X) W.$ 

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Theorem (Denisov, 1990; Carmeli, Heinosaari, Toigo, 2009)

 ${\mathcal I}$  is an U-covariant instrument based on  $\Omega$  iff there exists

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# Inducing functor and imprimitivity theorem

#### Basic assumptions

In the following

- *H* is a closed subgroup of *G*
- $\Omega$  is the homogeneous space G/H

#### Induction is a functor

 $\operatorname{Rep}(H) \ni \sigma \longmapsto (V^{\sigma}, \mathsf{P}^{\sigma}, \mathcal{K}^{\sigma}) \in \operatorname{Impr}(G, G/H)$ 

### Theorem (Mackey)

Induction is an equivalence of categories.

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Induction is an equivalence of categories.

General structure theorems Explicit structure theorems and examples

## General problem

For all  $\sigma \in \operatorname{Rep}(H)$ 

Diagonalize ( $V^{\sigma}, \mathcal{K}^{\sigma}$ )

## $\downarrow$

Find all possible isometries  $W: \mathcal{H} \to \mathcal{K}^{\sigma}$  such that  $WU(g) = V^{\sigma}(g)W \ \forall g \in G$ 

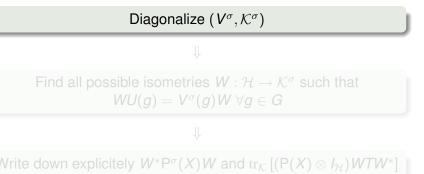
## $\downarrow$

Write down explicitly  $W^* P^{\sigma}(X) W$  and  $\operatorname{tr}_{\mathcal{K}}[(P(X) \otimes I_{\mathcal{H}}) WTW^*]$ 

General structure theorems Explicit structure theorems and examples

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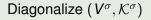
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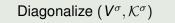
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# Particular solutions

A complete solution is available in the following cases

- (1) G generic, H compact, U irreducible
- (2) G abelian, H generic, U generic
- (3) G compact, H generic, U generic
- (4)  $G = H \ltimes K$ , H normal and abelian, U irreducible

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Phase-space observables and instruments  $\in$  (1)

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## Phase observables $\in$ (2) $\cap$ (3)

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### Example

Position and momentum observables  $\in$  (4)

Theorem (Cassinelli, De Vito, Toigo, 2003)

Suppose U projective and irreducible,  $H \subset G$  compact. Then there exists U-covariant POVMs on G/H iff U is square integrable.

In this case, there exists a selfadjoint positive operator  $C: \mathcal{H} \to \mathcal{H}$  such that every U-covariant POVM on G/H is of the form

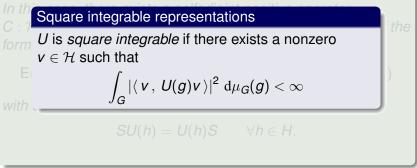
$$\mathsf{E}(X) = \int_X U(g) CSCU(g)^{-1} d\mu_{G/H}(gH) \qquad \forall X \in \mathcal{B}(G/H)$$

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$$SU(h) = U(h)S \quad \forall h \in H.$$

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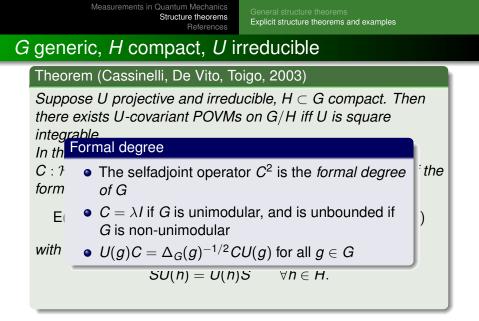
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 $\mathcal{I}_X^*(A) = \int_X U(g)C\Phi(U(g)^*AU(g))CU(g)^{-1} d\mu_{G/H}(gH)$ for all  $X \in \mathcal{B}(G/H)$ ,  $A \in \mathcal{L}(\mathcal{H})$ , where  $\Phi : \mathcal{L}(\mathcal{H}) o \mathcal{T}(\mathcal{H})$  is such that

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# Position and momentum observables

#### Theorem

E is a position observable iff there exists a probability density  $\rho$  on  $\mathbb R$  such that

$$\mathsf{E}(X) \equiv \mathsf{E}_{\rho}(X) := \int_{\mathbb{R}} \rho(X - x) \, d\mathsf{Q}(x)$$

#### Theorem

F is a position observable iff there exists a probability density  $\nu$  on  $\mathbb P$  such that

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#### Remark

The F is on  $\mathbb{P}$   $p_{S}^{E_{\rho}} = \rho * p_{S}^{Q}$   $p_{S}^{E_{\nu}} = \nu * p_{S}^{P}$  $F(Y) \equiv F_{\nu}(Y) := \int_{\mathbb{P}} \nu(Y - y) dP(y)$ 

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Measurements in Quantum Mechanics Structure theorems References General structure theorems Explicit structure theorems and examples

# Phase space

• There exists phase space observables, and every phase space observable is of the form

$$\mathsf{G}(X \times Y) = \frac{1}{2\pi} \iint_{X \times Y} U(a) V(p) S V(p)^* U(a)^* \, \mathrm{d}a \, \mathrm{d}p$$

with  $\boldsymbol{\mathcal{S}}\in\mathcal{S}(\mathcal{H})$ 

• The margins of G

$$\begin{array}{lll} \mathsf{E}(X) & := & \mathsf{G}(X \times \mathbb{P}) \\ \mathsf{F}(Y) & := & \mathsf{G}(\mathbb{R} \times Y) \end{array}$$

are position and momentum observables, respectively. Their associated probability densities  $\rho$ ,  $\nu$  are absolutely continuous wrt the Lebesgue measure and *Fourier-related* 

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Position and momentum observables  $E_{\rho}$ ,  $F_{\nu}$  are *coexistent* if they are margins of a single observable G on  $\mathbb{R} \times \mathbb{P}$ 

#### Theorem

 $\mathsf{E}_\rho$  and  $\mathsf{F}_\nu$  are coexistent iff they are margins of a phase space observable.

#### Consequences

If E<sub>ρ</sub> and F<sub>ν</sub> are coexistent, then ρ, ν are absolutely continuous wrt the Lebesgue measure and Fourier-related
 Var (ρ<sub>S</sub><sup>E<sub>ρ</sub></sup>) Var (ρ<sub>S</sub><sup>F<sub>ν</sub></sup>) ≥ 1 for all S ∈ S(H)

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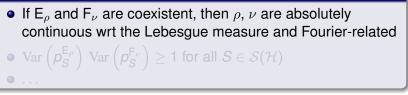
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# Position and momentum instruments

## Definition

An instrument  $\mathcal{I}: \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$  is a *position instrument* if

$$U(a)\mathcal{I}_X(U(a)^*TU(a))U(a)^* = \mathcal{I}_{X+a}(T)$$
  
$$V(p)\mathcal{I}_X(V(p)^*TV(p))V(p)^* = \mathcal{I}_X(T)$$

A similar definition holds for *momentum instruments*. In particular,  $\mathcal{J} : \mathcal{B}(\mathbb{P}) \to \mathcal{L}(\mathcal{T}(\mathcal{H}))$  is a momentum instrument iff

$$\mathcal{I}_X(T) = \mathcal{F}\mathcal{J}_X(\mathcal{F}^{-1}T\mathcal{F})\mathcal{F}^{-1}$$

defines a position instrument  $\mathcal{I}$ .

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#### Theorem

There is a one-to-one correspondence between position instruments and couples  $(\mu, K)$ , with

- $\mu$  probability measure on  $\mathbb R$
- $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$  positive semidefinite kernel satisfying

$$\int K(x,h;x,h)\,dx\,d\mu(h)=1$$

If  $(\mu, K)$  is as above, the corresponding position instrument is  $[\mathcal{I}_X(T)](x, y) = \int 1_X(z)T(x+h, y+h)K(z-y, h; z-x, h) dz d\mu(h)$ 

#### Theorem

There is a one-to-one correspondence between position instruments and couples  $(\mu, K)$ , with

- $\mu$  probability measure on  $\mathbb R$
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#### Theorem

There is a one-to-one correspondence between position instruments and couples  $(\mu, K)$ , with

i, j=1

A map  $K : X \times X \to \mathbb{C}$  is positive semidefinite if for all  $N \in \mathbb{Z}_+, \{x_i\}_{i=1,2...N} \subset X, \{c_i\}_{i=1,2...N} \subset \mathbb{C}$ 

$$\sum_{i=1}^{N} c_i \overline{c_i} K(x_i, x_i) \geq 0$$

If  $(\mu,$ 

$$[\mathcal{I}_{X}(T)](x,y) = \int 1_{X}(z) I(x+n,y+n)K(z-y,n;z-x,n) \, dz \, d\mu(h)]$$

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# Structure theorem for position instruments

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# Structure theorem for position instruments

#### Theorem

There is a one-to-one correspondence between position instruments and couples  $(\mu, K)$ , with

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- Remark

Trace class operators have been identified with their associated kernels, i. e.

If 
$$(\mu, \qquad Tf(x) = \int T(x, y)f(y) \, \mathrm{d}y \qquad \forall f \in L^2(\mathbb{R})$$

$$[\mathcal{I}_X(T)](x,y) = \int \mathbf{1}_X(z)T(x+h,y+h)K(z-y,h;z-x,h)\,dz\,d\mu(h)$$

is

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# G abelian, H generic, U generic

• Dual group:

$$\hat{\pmb{G}} = \{\gamma \in \pmb{\mathcal{C}}(\pmb{G}) \mid \gamma(\pmb{g}\pmb{h}) = \gamma(\pmb{g})\gamma(\pmb{h}) \; \forall \pmb{g}, \pmb{h} \in \pmb{G}\}$$

• Annihilator of *H* in  $\hat{G}$ :

$$H^{\perp} = \left\{ \gamma \in \hat{G} \mid \gamma(h) = 1 \ \forall h \in H \right\}$$

• Quotient map:

$$p: \hat{G} \to \hat{G}/H^{\perp}$$

We have identifications

$$\hat{H} = \hat{G}/H^{\perp}$$
 (canonical)  
 $\hat{G} = \hat{G}/H^{\perp} imes H^{\perp}$  (depending on a cross-section)

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#### Theorem

Let U be the diagonal representation in  $\mathcal{H}=\mathsf{L}^{\mathsf{2}}\left(\hat{\mathsf{G}},
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ight).$ 

There exists U-covariant POVMs on G/H iff  $\nu = \alpha p(\nu) \otimes \mu_{H^{\perp}}$ . In this case, every U-covariant POVM on G/H has the form

$$\langle E(X)\psi, \phi \rangle = \int_{\widehat{G}} d\nu(\gamma) \int_{H^{\perp}} d\mu_{H^{\perp}}(\chi) \int_{X} d\mu_{G/H}(gH) \sqrt{\alpha(\gamma)\alpha(\gamma\chi^{-1})} \, \chi(gH) \left\langle W(\gamma\chi^{-1})\psi(\gamma\chi^{-1}), W(\gamma)\phi(\gamma\chi^{-1}) \right\rangle$$

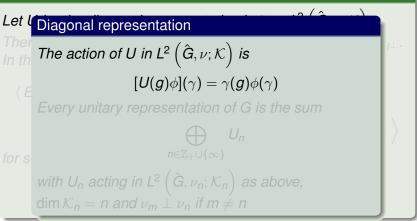
for some choice of a map

$$\hat{G} 
i \gamma \mapsto W_{\gamma} \in \mathcal{L}(\mathcal{K}; \mathcal{K}_{\infty}) \qquad W_{\gamma}^* W_{\gamma} = I_{\mathcal{K}}$$

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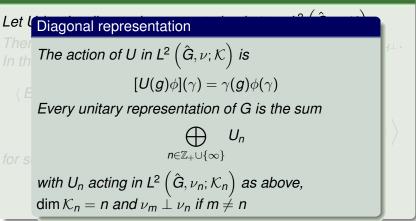
# *G* abelian, *H* generic, *U* generic

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General structure theorems Explicit structure theorems and examples

#### Example: phase observables

- $G = \mathbb{T}, H = \{1\}$
- *H* = *L*<sup>2</sup> (ℕ, #)
- $U(z)\delta_n = z^n \delta_n$

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Every phase observable is of the form

$$\mathsf{E}(X) = \sum_{m,n\in\mathbb{N}} c_{mn} \int_X z^{m-n} \, \mathrm{d} z \, |m\rangle \, \langle n| \qquad \forall X \in \mathcal{B}(\mathbb{T})$$

where the matrix  $\{c_{mn} = \langle \eta_n, \eta_m \rangle\}_{m,n \in \mathbb{N}}$  is positive semidefinite and  $c_{nn} = 1$ .

In particular, there is no sharp phase observable.

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General structure theorems Explicit structure theorems and examples

# G compact, H compact, U generic

$$egin{aligned} \mathcal{H} &= igoplus_{\pi \in \hat{G}} \mathcal{H}_{\pi} \otimes \mathcal{K}_{\pi} \ U &= igoplus_{\pi \in \hat{G}} \pi \otimes I_{\mathcal{K}_{\pi}} \end{aligned}$$

where  $\hat{G}$  is the set of irreducible unitary representations  $(\pi, \mathcal{H}_{\pi})$  of *G* 

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Let  $\mathcal{C}$  be the convex set of maps  $K : \hat{G} \times \hat{G} \rightarrow \mathcal{L}(\mathcal{H})$  s. t.

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$$K(\rho,\pi)U(h) = U(h)K(\rho,\pi)$$
 for all  $h \in H$ 

2  $\sum_{\rho,\pi} \langle K(\rho,\pi) v^{\pi}, v^{\rho} \rangle \ge 0$  for all sequences  $\{v^{\pi}\}_{\pi \in \hat{G}}$  in  $\mathcal{H}$  with only a finite number of nonzero elements

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#### Theorem

There is a one-to-one convex correspondence betwteen C and the set of U-covariant POVMs on G/H, given by

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for all  $\mathsf{v}_{\pi} \in \mathcal{H}_{\pi} \otimes \mathcal{K}_{\pi}$ ,  $\mathsf{w}_{
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- The POVMs on a set Ω form a convex set P(Ω), which is compact in a suitable weak-\* topology
- If E ∉ ext P(Ω), then E = p<sub>1</sub>E<sub>1</sub> + p<sub>2</sub>E<sub>2</sub> with p<sub>1</sub> + p<sub>2</sub> = 1,
   i. e. E comes from the random choice between two different observables
- Every  $E \in \mathcal{P}(\Omega)$  can be approximated by the convex sum of elements in ext  $\mathcal{P}(\Omega)$
- The elements in ext P(Ω) optimize convex cost functions in quantum estimation theory

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# An application: extremal POVMs

#### For $K \in C$ , let $\mathcal{H}_K$ be the RKHS associated to K.

 $\mathcal{H}_{\mathcal{K}}$  carries an unitary representation  $\tilde{U}$  of H, given by  $[\tilde{U}(h)f](\pi) = U(h)f(\pi)$ 

$$\begin{aligned} \mathcal{T}_U &:= \{ T \in \mathcal{T}(\mathcal{H}) \mid TU(g) = U(g)T \; \forall g \in G \} \\ \mathcal{T}_{\tilde{U}} &:= \left\{ T \in \mathcal{T}(\mathcal{H}_K) \mid T\tilde{U}(h) = \tilde{U}(h)T \; \forall h \in H \right\} \\ \tilde{\mathcal{T}}_U &:= \overline{\operatorname{span}} \; \left\{ \operatorname{ev}_\pi^* T \operatorname{ev}_\pi \mid T \in \mathcal{T}_U, \; \pi \in \hat{G} \right\} \end{aligned}$$

# An application: extremal POVMs

For  $K \in C$ , let  $\mathcal{H}_K$  be the RKHS associated to K.

 $\mathcal{H}_{\mathcal{K}} \in \mathbb{R} \text{eproducing kernel Hilbert space associated to } \mathcal{K}$   $\bullet \ \mathcal{H}_{\mathcal{K}}^{0} = \operatorname{span} \left\{ \mathcal{K}(\cdot, \pi) \mathcal{V} \mid \pi \in \hat{\mathcal{G}}, \ \mathcal{V} \in \mathcal{H} \right\}$   $\bullet \ \langle \cdot, \cdot \rangle_{\mathcal{K}} : \mathcal{H}_{\mathcal{K}}^{0} \times \mathcal{H}_{\mathcal{K}}^{0} \to \mathbb{C} \text{ given by}$   $\langle \mathcal{K}(\cdot, \pi) \mathcal{V}, \ \mathcal{K}(\cdot, \rho) \mathcal{W} \rangle_{\mathcal{K}} = \langle \mathcal{K}(\rho, \pi) \mathcal{V}, \ \mathcal{W} \rangle$ 

Then the completion  $\mathcal{H}_K$  of  $\mathcal{H}_K^0$  is a Hilbert space of  $\mathcal{H}$ -valued functions in which the evaluation maps  $ev_\pi$  are continuous

## An application: extremal POVMs

For  $K \in C$ , let  $\mathcal{H}_K$  be the RKHS associated to K.

 $\mathcal{H}_{\mathcal{K}} \stackrel{\mathsf{Reproducing kernel Hilbert space associated to } \mathcal{K}$ •  $\mathcal{H}_{\mathcal{K}}^{0} = \operatorname{span} \left\{ \mathcal{K}(\cdot, \pi) \mathcal{V} \mid \pi \in \hat{\mathcal{G}}, \ \mathcal{V} \in \mathcal{H} \right\}$ •  $\langle \cdot, \cdot \rangle_{\mathcal{K}} : \mathcal{H}_{\mathcal{K}}^{0} \times \mathcal{H}_{\mathcal{K}}^{0} \to \mathbb{C} \text{ given by}$  $\langle \mathcal{K}(\cdot, \pi) \mathcal{V}, \ \mathcal{K}(\cdot, \rho) \mathcal{W} \rangle_{\mathcal{K}} = \langle \mathcal{K}(\rho, \pi) \mathcal{V}, \ \mathcal{W} \rangle$ 

Then the completion  $\mathcal{H}_K$  of  $\mathcal{H}_K^0$  is a Hilbert space of  $\mathcal{H}$ -valued functions in which the evaluation maps  $ev_\pi$  are continuous

# An application: extremal POVMs

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# An application: extremal POVMs

For  $K \in C$ , let  $\mathcal{H}_K$  be the RKHS associated to K.

# $\mathcal{H}_{\mathcal{K}}$ carries an unitary representation $\tilde{U}$ of H, given by $[\tilde{U}(h)f](\pi) = U(h)f(\pi)$

$$\mathcal{T}_U := \{ T \in \mathcal{T}(\mathcal{H}) \mid TU(g) = U(g)T \ \forall g \in G \}$$
  
 $\mathcal{T}_{\tilde{U}} := \left\{ T \in \mathcal{T}(\mathcal{H}_K) \mid T\tilde{U}(h) = \tilde{U}(h)T \ \forall h \in H \right\}$   
 $\tilde{\mathcal{T}}_U := \overline{\operatorname{span}} \ \left\{ \operatorname{ev}_{\pi}^* T \operatorname{ev}_{\pi} \mid T \in \mathcal{T}_U, \ \pi \in \hat{G} \right\}$ 

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# An application: extremal POVMs

#### Theorem

The POVM associated to K is extremal in the set of U-covariant POVMs on G/H iff  $\tilde{T}_U = T_{\tilde{U}}$ .

#### Example: extremal phase observables

Let E be a phase observable with associated matrix  $\{c_{mn}\}_{m,n\in\mathbb{N}}$ . Let

- $\{\eta_n\}_{n\in\mathbb{N}}$  be vectors in  $\mathcal{H}$  such that  $c_{mn} = \langle \eta_m, \eta_n \rangle$
- $\mathcal{H}_0$  be their closed linear span.

Then E is extremal in the convex set of phase observables iff

 $\mathcal{T}(\mathcal{H}_0) = \overline{\operatorname{span}} \{ |\eta_n\rangle \langle \eta_n| \mid n \in \mathbb{N} \}$ 

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### Indice

#### Measurements in Quantum Mechanics

- Observables and instruments
- Symmetry groups and covariant measurements
- Examples

#### 2 Structure theorems

- General structure theorems
- Explicit structure theorems and examples

# 3 References

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