Normal completely positive maps on the space of quantum operations

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Mathematical setting

- $\mathcal{M} \subset \mathcal{L}(\mathcal{H}), \mathcal{N} \subset \mathcal{L}(\mathcal{K}) \ldots$: complex and separable von Neumann algebras
- $\mathcal{M}_+, \mathcal{N}_+, \ldots$: cones of their positive elements
- $M_n := \mathcal{L}(\mathbb{C}^n)$: von Neumann algebra of complex $n \times n$-matrices
- $\mathcal{M} \hat{\otimes} \mathcal{N}$: algebraic tensor product of $\mathcal{M}$ and $\mathcal{N}$
- $\mathcal{M} \bar{\otimes} \mathcal{N}$: von Neumann algebra tensor product of $\mathcal{M}$ and $\mathcal{N}$

Recall that

$$\mathcal{M} \hat{\otimes} \mathcal{N} \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$$

and

$$\mathcal{M} \bar{\otimes} \mathcal{N} = \text{weak*-closure of } \mathcal{M} \hat{\otimes} \mathcal{N}$$

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Mathematical setting

Remarks

- $\mathcal{L}(\mathcal{H})\bar{\otimes}\mathcal{L}(\mathcal{K}) = \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$
- If $\dim \mathcal{N} < \infty$, then $\mathcal{M}\hat{\otimes}\mathcal{N} = \mathcal{M}\bar{\otimes}\mathcal{N}$
- If $\dim \mathcal{N}_i < \infty$ and $\mathcal{E} : \mathcal{M}_1 \to \mathcal{M}_2$, $\mathcal{F} : \mathcal{N}_1 \to \mathcal{N}_2$ are linear maps, then $\mathcal{E} \otimes \mathcal{F} : \mathcal{M}_1\bar{\otimes}\mathcal{N}_1 \to \mathcal{M}_2\bar{\otimes}\mathcal{N}_2$ is well-defined

Definition

A linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is *completely positive* if the map

$$\mathcal{E} \otimes \mathcal{I}_n : \mathcal{M}\bar{\otimes}M_n \to \mathcal{N}\bar{\otimes}M_n$$

is positive for all $n \in \mathbb{N}$.
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Definition

A linear map $\mathcal{E}$: ... ap

It then makes sense to speak about positivity and boundedness of $\mathcal{E}\otimes\mathcal{F}$

$$\mathcal{E}\otimes\mathcal{I}_n : \mathcal{M}\otimes M_n \to \mathcal{N}\otimes M_n$$

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Definition of quantum operations and channels

Definition

A linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is a *quantum operation* if it is

(i) completely positive

(ii) normal:

$$\mathcal{E}(A_n) \uparrow \mathcal{E}(A) \text{ for all sequences } \{A_n\}_{n \in N} \text{ in } \mathcal{M}_+ \text{ s. t. } A_n \uparrow A$$

(iii) subnormalized:

$$\mathcal{E}(I_{\mathcal{M}}) \leq I_{\mathcal{N}}$$

$\mathcal{E}$ is a *quantum channel* if condition (iii) is replaced by

(iii') normalized:

$$\mathcal{E}(I_{\mathcal{M}}) = I_{\mathcal{N}}$$

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Notations

- $\mathrm{CP}\,(\mathcal{M}, \mathcal{N})$: normal completely positive maps in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{N})$
- $\mathrm{CP}_0\,(\mathcal{M}, \mathcal{N})$: subset of quantum operations
- $\mathrm{CP}_1\,(\mathcal{M}, \mathcal{N})$: subset of quantum channels

Clearly,

$$\mathrm{CP}_1\,(\mathcal{M}, \mathcal{N}) \subset \mathrm{CP}_0\,(\mathcal{M}, \mathcal{N}) \subset \mathrm{CP}\,(\mathcal{M}, \mathcal{N})$$

(iii') normalized:

$$\mathcal{E}(I_{\mathcal{M}}) = I_{\mathcal{N}}$$

Stinespring Theorem

Theorem

Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{N} \subset \mathcal{L}(\mathcal{K})$. *A linear map* $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ *is a quantum operation iff there exist a Hilbert space* $\mathcal{V}$ *and a bounded operator* $V : \mathcal{K} \to \mathcal{H} \otimes \mathcal{V}$, *with* $\|V\|_\infty \leq 1$, *such that*

$$\mathcal{E}(A) = V^*(A \otimes I_{\mathcal{V}})V \qquad \forall A \in \mathcal{M}.$$

In this case, $\mathcal{E}$ *is a quantum channel iff* $V^* V = I_{\mathcal{K}}$

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Statement of the problem

Problem

Characterize the transformations

$$\mathsf{S} : \mathrm{CP}_0\left(\mathcal{M}, \mathcal{N}\right) \to \mathrm{CP}_0\left(\mathcal{M}, \mathcal{N}\right)$$

or, more generally,

$$\mathsf{S} : \mathrm{CP}_0\left(\mathcal{M}_1, \mathcal{N}_1\right) \to \mathrm{CP}_0\left(\mathcal{M}_2, \mathcal{N}_2\right)$$

which are admissible in Quantum Mechanics (*quantum supermaps*)

Applications

- Quantum information
- Quantum measurement theory
- Quantum process tomography

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Properties of supermaps: Linearity

A quantum supermap must preserve mixtures and $\mathsf{S}(0) = 0$

$\Downarrow$

S is convex and $\mathsf{S}(0) = 0$

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S uniquely extends to a linear map defined on $\mathrm{span}\,\mathrm{CP}_0\,(\mathcal{M},\mathcal{N})$

But what is the linear space spanned by $\mathrm{CP}_0\,(\mathcal{M},\mathcal{N})$?

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Properties of supermaps: Linearity

Example

Suppose

$$\mathcal{M} = \mathcal{N} = \mathcal{L}(\mathcal{H}).$$

- If $\mathcal{H} = \mathbb{C}^n$

$$\begin{aligned} \text{span}\, \text{CP}_0\,(\mathcal{M}, \mathcal{N}) &= \text{span}\, \text{CP}_0\,(M_n, M_n) \\ &= M_{n^2} \\ &= \text{Hom}_{\mathbb{C}}(M_n, M_n) \end{aligned}$$

- If $\dim \mathcal{H} = \infty$

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$$\operatorname{span} \mathrm{CP}_0 \left(\mathcal{M}, \mathcal{N} \right) = \operatorname{span} \mathrm{CP}_0 \left(M_n, M_n \right)$$

By Choi isomorphism,

$$\mathrm{CP} \left(M_n, M_n \right) \simeq M_{n^2 +}$$

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Completely bounded maps: Definition

Definition

A linear map $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is *completely bounded* if $\exists\ C > 0$ such that

$$\|(\mathcal{E} \otimes \mathcal{I}_n)(A)\|_\infty \leq C \|A\|_\infty$$

for all $A \in \mathcal{M} \bar{\otimes} M_n$ and $n \in \mathbb{N}$

Example

Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{N} = \mathcal{L}(\mathcal{K})$. If $E, F \in \mathcal{L}(\mathcal{K}; \mathcal{H})$, the map

$$\begin{array}{rccc} E^* \odot_{\mathcal{M}} F : & \mathcal{M} & \longrightarrow & \mathcal{L}(\mathcal{K}) \\ & A & \longmapsto & E^* A F \end{array}$$

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Notation

$\mathrm{CB}\left(\mathcal{M}, \mathcal{N}\right)$: completely bounded weak*-continuous maps from $\mathcal{M}$ to $\mathcal{N}$

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Completely bounded maps: Properties

1. **Ordering:** $\mathrm{CP}(\mathcal{M}, \mathcal{N})$ is a cone in $\mathrm{CB}(\mathcal{M}, \mathcal{N})$, hence it induces a linear ordering in $\mathrm{CB}(\mathcal{M}, \mathcal{N})$, which we denote by $\preceq$
2. **Tensoring:** If $\mathcal{E} \in \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1)$, $\mathcal{F} \in \mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$, the product
$$\mathcal{E} \otimes \mathcal{F} : \mathcal{M}_1 \hat{\otimes} \mathcal{N}_1 \longrightarrow \mathcal{M}_2 \hat{\otimes} \mathcal{N}_2$$
uniquely extends to a map
$$\mathcal{E} \otimes \mathcal{F} \in \mathrm{CB}(\mathcal{M}_1 \bar{\otimes} \mathcal{M}_2, \mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)$$
3. **Spanning:** If $\mathcal{N} = \mathcal{L}(\mathcal{K})$, every $\mathcal{E} \in \mathrm{CB}(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ can be written
$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2 + i(\mathcal{E}_3 - \mathcal{E}_4).$$
for some $\mathcal{E}_1, \ldots, \mathcal{E}_4 \in \mathrm{CP}(\mathcal{M}, \mathcal{L}(\mathcal{K}))$. In particular,
$$\mathrm{span}\,\mathrm{CP}_0(\mathcal{M}, \mathcal{L}(\mathcal{K})) = \mathrm{CB}(\mathcal{M}, \mathcal{L}(\mathcal{K}))$$

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Remark

If $\mathcal{M} \subset M_m$ and $\mathcal{N} \subset M_n$, then

$$\mathrm{CB}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}, \mathcal{N}).$$

Properties (1) and (2) are trivial, and (3) follows from Choi isomorphism

Remarks

- For composite systems,

$$\mathrm{CB}\left(\mathcal{M}\bar{\otimes}M_m,\mathcal{N}\bar{\otimes}M_n\right) = \mathrm{CB}\left(\mathcal{M},\mathcal{N}\right)\hat{\otimes}\mathrm{Hom}_{\mathbb{C}}(M_m,M_n)$$
$$= \mathrm{CB}\left(\mathcal{M},\mathcal{N}\right)\hat{\otimes}M_{mn}$$

- If

$$\mathsf{S} : \mathrm{CB}\left(\mathcal{M}_1,\mathcal{N}_1\right) \to \mathrm{CB}\left(\mathcal{M}_2,\mathcal{N}_2\right)$$
$$\mathsf{T} : \mathrm{Hom}_{\mathbb{C}}(M_{m_1},M_{n_1}) \to \mathrm{Hom}_{\mathbb{C}}(M_{m_2},M_{n_2})$$

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A quantum supermap must preserve quantum operations on composite systems

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Here, $\mathsf{I}_n : \mathrm{Hom}_{\mathbb{C}}(M_n, M_n) \to \mathrm{Hom}_{\mathbb{C}}(M_n, M_n)$ is the identity map

In other words, S is completely positive if $\mathsf{S} \otimes \mathsf{I}_n$ preserves the linear ordering $\preceq$ for all $n \in \mathbb{N}$

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A linear map $\mathsf{S} : \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{N}_2)$ is *completely positive* if

$$(\mathsf{S} \otimes \mathsf{I}_n)(\mathrm{CP}(\mathcal{M}_1 \bar{\otimes} M_n, \mathcal{N}_1 \bar{\otimes} M_n)) \subset \mathrm{CP}(\mathcal{M}_2 \bar{\otimes} M_n, \mathcal{N}_2 \bar{\otimes} M_n) \qquad \forall n \in \mathbb{N}$$

Here, $\mathsf{I}_n : \mathrm{Hom}_{\mathbb{C}}(M_n, M_n) \to \mathrm{Hom}_{\mathbb{C}}(M_n, M_n)$ is the identity map

In other words, S is completely positive if $\mathsf{S} \otimes \mathsf{I}_n$ preserves the linear ordering $\preceq$ for all $n \in \mathbb{N}$

Properties of supermaps: Normality

A quantum supermap $\mathsf{S} : \mathrm{CB}\left(\mathcal{M}_1, \mathcal{N}_1\right) \to \mathrm{CB}\left(\mathcal{M}_2, \mathcal{N}_2\right)$ must be continuous in a suitable sense

…But, if $\dim \mathcal{M}_i = \infty$ or $\dim \mathcal{N}_i = \infty$, many 'natural' topologies are available on the space $\mathrm{CB}\left(\mathcal{M}, \mathcal{N}\right)$ (e. g. pointwise uniform, strong, weak* or weak convergence…)

…So we avoid reference to a particular topology, and require normality with respect to a single suitable notion of increasing sequences in $\mathrm{CB}\left(\mathcal{M}, \mathcal{N}\right)$

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Increasing sequences in $\mathrm{CB}(\mathcal{M}, \mathcal{N})$

Definition

A sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in $\mathrm{CB}(\mathcal{M}, \mathcal{N})$ is

- *CP-increasing* if $0 \preceq \mathcal{E}_m \preceq \mathcal{E}_n$ for $m \leq n$
- *CP-bounded* if $\exists \mathcal{F} \in \mathrm{CP}(\mathcal{M}, \mathcal{N})$ such that $\mathcal{E}_n \preceq \mathcal{F}$ for all n

Proposition

If the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in $\mathrm{CB}(\mathcal{M}, \mathcal{N})$ is CP-increasing and CP-bounded, then $\exists!\ \mathcal{E} \in \mathrm{CP}(\mathcal{M}, \mathcal{N})$ such that

$$\operatorname*{wk^*\text{-}lim}_{n\to\infty} \mathcal{E}_n(A) = \mathcal{E}(A) \qquad \forall A \in \mathcal{M}$$

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$$\mathcal{E}_n \Uparrow \mathcal{E}$$

Example (Kraus Theorem)

Suppose $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{N} = \mathcal{L}(\mathcal{K})$. For all $\mathcal{E} \in \mathrm{CP}(\mathcal{M}, \mathcal{L}(\mathcal{K}))$ $\exists$ a sequence $\{E_k\}_{k\in\mathbb{N}}$ in $\mathcal{L}(\mathcal{K}, \mathcal{H})$ such that

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A linear map

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is *normal* if for all sequences $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ we have

$$\mathcal{E}_n \Uparrow \mathcal{E} \quad \text{implies} \quad \mathsf{S}(\mathcal{E}_n) \Uparrow \mathsf{S}(\mathcal{E})$$

Remark

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is a *quantum supermap* if

(i) S is completely positive

(ii) S is normal.

Definition

A quantum supermap S is *deterministic* if

$$\mathsf{S}(\mathrm{CP}_1\left(\mathcal{M}_1, \mathcal{N}_1\right)) \subset \mathrm{CP}_1\left(\mathcal{M}_2, \mathcal{N}_2\right)$$

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Definition of quantum supermaps

The set of quantum supermaps is ordered: given two quantum supermaps S, T acting in the same $\mathcal{CB}$ spaces, we will write

$$\mathsf{S} \ll \mathsf{T} \quad \text{iff} \quad \mathsf{T} - \mathsf{S} \text{ is a quantum supermap}$$

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A quantum supermap S is *probabilistic* if there exists a deterministic quantum supermap T such that

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Examples

Example: Amplification

If $\mathcal{V}$ is a Hilbert space, then

$$\begin{array}{rccc} \Pi_{\mathcal{V}}: & \mathrm{CB}\,(\mathcal{M},\mathcal{N}) & \longrightarrow & \mathrm{CB}\,(\mathcal{M}\bar{\otimes}\mathcal{L}(\mathcal{V}),\mathcal{N}\bar{\otimes}\mathcal{L}(\mathcal{V})) \\ & \mathcal{E} & \longmapsto & \mathcal{E}\otimes\mathcal{I}_{\mathcal{V}} \end{array}$$

is a deterministic quantum supermap

Example: Concatenation

If $\mathcal{A} \in \mathrm{CP}_0\,(\mathcal{N}_1,\mathcal{N}_2)$, $\mathcal{B} \in \mathrm{CP}_0\,(\mathcal{M}_2,\mathcal{M}_1)$, then

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Dilation of deterministic supermaps

Theorem (Dilation theorem)

A linear map

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is a deterministic supermap iff there exist a triple $(\mathcal{V}, V, \mathcal{F})$*, where*

- $\mathcal{V}$ *is a Hilbert space*
- $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$ *is an isometry*
- $\mathcal{F} : \mathcal{M}_2 \to \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V})$ *is a quantum channel*

such that

$$[\mathsf{S}(\mathcal{E})](A) = V^* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}) \mathcal{F}(A) \right] V$$

for all $\mathcal{E} \in \mathrm{CB}\left(\mathcal{M}_1, \mathcal{L}\left(\mathcal{K}_1\right)\right)$ *and* $A \in \mathcal{M}_2$

Dilation of deterministic supermaps

1. The triple $(\mathcal{V}, V, \mathcal{F})$ can always be chosen in a way that
$$\mathcal{V} = \overline{\text{span}} \{(u^* \otimes I_{\mathcal{V}})Vv \mid u \in \mathcal{K}_1 , v \in \mathcal{K}_2\}$$
(*minimal dilation*)

2. In this case, if $(\mathcal{V}', V', \mathcal{F}')$ is another dilation, then $\exists!$ isometry $W : \mathcal{V} \longrightarrow \mathcal{V}'$ such that
$$V' = (I_{\mathcal{K}_1} \otimes W)V$$
and
$$\mathcal{F}(A) = (I_{\mathcal{M}_1} \otimes W^*)\mathcal{F}'(A)(I_{\mathcal{M}_1} \otimes W) \quad \forall A \in \mathcal{M}_2$$

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Dilation of deterministic supermaps

3. Setting

$$\mathcal{A} = V^* \odot_{\mathcal{L}(\mathcal{K}_1 \otimes \mathcal{V})} V$$

we have that S is the composition

$$\mathsf{S} = \mathsf{C}_{\mathcal{A},\mathcal{F}} \circ \Pi_{\mathcal{V}}$$

4. In the Schrödinger (predual) picture

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \mathcal{F}_* \left[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_* (V \rho V^*) \right]$$

for all $\mathcal{E} \in \mathrm{CB}\left(\mathcal{M}_1, \mathcal{L}\left(\mathcal{K}_1\right)\right)$ and $\rho \in \mathcal{L}\left(\mathcal{K}_2\right)_* = \mathcal{T}\left(\mathcal{K}_2\right)$

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Dilation of deterministic supermaps

5. Taking $\mathcal{M}_1 = \mathcal{M}_2 = \mathbb{C}$, one gets Stinespring Theorem for normal CP maps
6. Taking $\mathcal{K}_1 = \mathcal{K}_2 = \mathbb{C}$, one gets Stinespring Theorem for normal CP maps in the Schrödinger picture

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Dilation of probabilistic supermaps

Theorem (Radon-Nikodym theorem for supermaps)

Suppose $\mathsf{S}, \mathsf{T} : \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)) \to \mathrm{CB}(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$ *are quantum supermaps, with*

$$\mathsf{T} \ll \mathsf{S}\,.$$

Suppose S *is deterministic, and let* $(\mathcal{V}, V, \mathcal{F})$ *be its minimal dilation. Then* $\exists!\, \mathcal{G} \in \mathrm{CP}_0(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V}))$*, with*

$$\mathcal{G} \preceq \mathcal{F},$$

such that

$$[\mathsf{T}(\mathcal{E})](A) = V^* [(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}}) \mathcal{G}(A)] V$$

for all $\mathcal{E} \in \mathrm{CB}(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$ *and* $A \in \mathcal{M}_2$

An example: Transforming POVMs into channels

- X : space of outcomes (discrete)
- $\mathcal{M}_1 = \ell^\infty(X)$: complex bounded functions (sequences) on X
- $\mathrm{CP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$: $\mathcal{L}(\mathcal{K}_1)$-valued POVMs on X

Let $(\mathcal{V}, V, \mathcal{F})$ be a dilation of a deterministic supermap

$$\mathsf{S} : \mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1)) \longrightarrow \mathrm{CB}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{K}_2))$$

For $x \in X$, set

$$\begin{array}{rccc} \mathcal{F}_{x\,*} : & \mathcal{T}(\mathcal{V}) & \longrightarrow & \mathcal{T}(\mathcal{H}_2) \\ & \sigma & \longmapsto & \mathcal{F}_*(\delta_x \sigma) \end{array}$$

Then

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If $\mathcal{E} \in \mathrm{CP}_1(\ell^\infty(X), \mathcal{L}(\mathcal{K}))$, then the map

$$P : \begin{array}{ccc} X & \longrightarrow & \mathcal{L}(\mathcal{K}) \\ x & \longmapsto & \mathcal{E}(\delta_x) \end{array}$$

is a POVM, and

$$\mathcal{E}(f) = \sum_{x \in X} f_x P_x$$

$x \in X$

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Note that we have

$$\mathcal{F} \in \mathrm{CP}_1(\mathcal{L}(\mathcal{H}_2), \ell^\infty(X)\bar{\otimes}\mathcal{L}(\mathcal{V}))$$

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$$\mathcal{F}_* : \ell^1(X; \mathcal{T}(\mathcal{V})) \longrightarrow \mathcal{T}(\mathcal{H}_2)$$

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> Transform the input system $\mathcal{K}_2$ into the composite system $\mathcal{K}_1 \otimes \mathcal{V}$

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Measure the POVM $\mathcal{E}$ on $\mathcal{K}_1$, thus obtaining the outcome $x \in X$

An example: Transforming POVMs into channels

- X : space of outcomes (discrete)
- $\mathcal{M}_1 = \ell^\infty(X)$: complex bounded functions (sequences) on X
- $\mathrm{CP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$: $\mathcal{L}(\mathcal{K}_1)$-valued POVMs on X

Let $(\mathcal{V}, V, \mathcal{F})$ be a dilation of a deterministic supermap

$$\mathsf{S} : \mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K}_1)) \longrightarrow \mathrm{CB}(\mathcal{L}(\mathcal{H}_2), \mathcal{L}(\mathcal{K}_2))$$

For $x \in X$, set

$$\mathcal{F}_{x\,*} : \begin{array}{ccc} \mathcal{T}(\mathcal{V}) & \longrightarrow & \mathcal{T}(\mathcal{H}_2) \\ \sigma & \longmapsto & \mathcal{F}_*(\delta_x \sigma) \end{array}$$

Then

$$[\mathsf{S}(\mathcal{E})]_*(\rho) = \sum_{x \in X} \mathcal{F}_{x\,*}[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*(V \rho V^*)_x]$$

Conditionally on the outcome x, apply the channel $\mathcal{F}_{x\,*}$ on the ancilla $\mathcal{V}$, thus converting it into the output $\mathcal{H}_2$

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Motivation

A quantum superinstrument R describes a measurement process of quantum channels

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$$\begin{array}{c} \mathcal{E} : \text{quantum channel} \\ \downarrow \\ \mathsf{R}(\mathcal{E}) \end{array}$$

Motivation

A quantum superinstrument R describes a measurement process of quantum channels

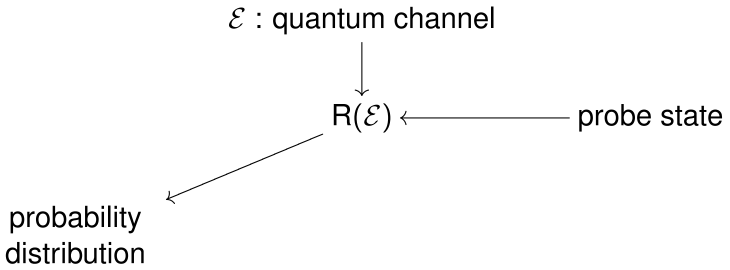
$\mathcal{E}$: quantum channel

$\downarrow$

R($\mathcal{E}$) $\longleftarrow$ probe state

Motivation

A quantum superinstrument R describes a measurement process of quantum channels



Motivation

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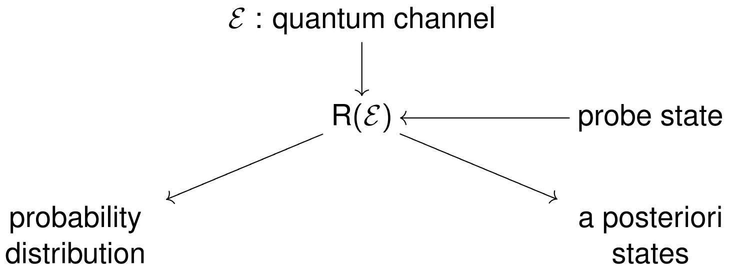

Motivation

- $\mathrm{CP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$: channels to be measured
- $\mathcal{L}(\mathcal{K}_2)_* = \mathcal{T}(\mathcal{K}_2)$: initial probe states
- $\mathcal{M}_{2*}$: final probe states
- $(\Omega, \mathcal{A})$: measurable set of outcomes
- $\mathcal{M}(\Omega; \mathrm{CP}_0(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2)))$: quantum instruments on the probes

A quantum superinstrument is a map

$$\mathrm{R} : \mathrm{CP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1)) \longrightarrow \mathcal{M}(\Omega; \mathrm{CP}_0(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2)))$$

To each channel $\mathcal{E} \in \mathrm{CP}_1(\mathcal{M}_1, \mathcal{L}(\mathcal{K}_1))$ it associates the instrument

$$\mathcal{A} \ni B \longmapsto \mathrm{R}_B(\mathcal{E}) \in \mathrm{CP}_0(\mathcal{M}_2, \mathcal{L}(\mathcal{K}_2))$$

- $\mathrm{CP}_1\left(\mathcal{M}_1, \mathcal{L}\left(\mathcal{K}_1\right)\right)$: channels to be measured
- $\mathcal{L}\left(\mathcal{K}_2\right)_* = \mathcal{T}\left(\mathcal{K}_2\right)$: initial probe states

Recall that (in the Heisenberg picture) a classical quantum instrument is just a map

$$\mathcal{J} : \mathcal{A} \to \mathrm{CP}_0\left(\mathcal{M}, \mathcal{L}(\mathcal{K})\right)$$

which is

- weak*-additive:

$$\mathcal{J}_B(A) = \underset{n\to\infty}{\mathrm{wk*\text{-}lim}} \sum_{i=1}^{n} \mathcal{J}_{B_i}(A) \quad \forall A \in \mathcal{M} \quad \text{if } B_i \cap B_j = \emptyset \ \forall i \neq j$$

- normalized:

$$\mathcal{J}_\Omega(I_\mathcal{M}) = I_\mathcal{K}$$

Motivation

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Definition of superinstruments

Definition

Let $(\Omega, \mathcal{A})$ be a measurable space. Suppose R is a map

$$\begin{array}{rccc} \mathsf{R}: & \mathcal{A} & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(\mathrm{CB}\,(\mathcal{M}_1, \mathcal{N}_1)\,;\mathrm{CB}\,(\mathcal{M}_2, \mathcal{N}_2)) \\ & B & \longmapsto & \mathsf{R}_B \end{array}$$

We say that R is a *quantum superinstrument* if

(i) R_B is a quantum supermap for all $B \in \mathcal{A}$

(ii) R_Ω is deterministic

(iii) if $B = \bigcup_{i=1}^{\infty} B_i$ with $B_i \cap B_j = \emptyset$ for $i \neq j$, then

$$[\mathsf{R}_B(\mathcal{E})](A) = \operatorname*{wk^*-lim}_{n\to\infty} \sum_{i=1}^{n} [\mathsf{R}_{B_i}(\mathcal{E})](A)$$

for all $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{N}_1)$ and $A \in \mathcal{M}_2$

Dilation of superinstruments

Theorem (Dilation of quantum superinstruments)

Suppose that

$$\mathsf{R} : \mathcal{A} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}\,(\mathcal{K}_1))\,; \mathrm{CB}\,(\mathcal{M}_2, \mathcal{L}\,(\mathcal{K}_2)))$$

is a quantum superinstrument. Then there exist a Hilbert space $\mathcal{V}$, an isometry $V : \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{V}$ and a quantum instrument

$$\mathcal{J} \in \mathcal{M}\,(\Omega; \mathrm{CP}_0\,(\mathcal{M}_2, \mathcal{M}_1 \bar{\otimes} \mathcal{L}(\mathcal{V})))$$

such that

$$[\mathsf{R}_B(\mathcal{E})](A) = V^*[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})\mathcal{J}_B(A)]V \quad \forall A \in \mathcal{M}_2$$

for all $B \in \mathcal{A}$ and $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}\,(\mathcal{K}_1))$

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such that (in the Schrödinger picture)

$$[\mathsf{R}_B(\mathcal{E})]_*(\rho) = \mathcal{J}_{B\,*}[(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})(V \rho V^*)] \quad \forall \rho \in \mathcal{T}\,(\mathcal{K}_2)$$

for all $B \in \mathcal{A}$ and $\mathcal{E} \in \mathrm{CB}\,(\mathcal{M}_1, \mathcal{L}\,(\mathcal{K}_1))$

An example: Quantum testers

- $\mathrm{CP}_1\left(\ell^\infty(X), \mathcal{L}(\mathcal{K})\right)$: input space of $\mathcal{L}(\mathcal{K})$-valued POVMs on X
- $\mathbb{C}$: trivial output space

Fix a quantum superinstrument

$$\mathsf{R} : \mathcal{A} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{CB}\left(\ell^\infty(X), \mathcal{L}(\mathcal{K})\right); \mathbb{C})$$

By dilation theorem

$$\mathsf{R}_B(\mathcal{E}) = \langle v, (\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})(\mathcal{J}_B) v\rangle \quad \forall \mathcal{E} \in \mathrm{CB}\left(\ell^\infty(X), \mathcal{L}(\mathcal{K})\right), \ B \in \mathcal{A}$$

For $x \in X$, define the $\mathcal{L}(\mathcal{V})$-valued POVM on Ω

$$Q_x : \mathcal{A} \to \mathcal{L}(\mathcal{V}) \qquad Q_{x\,B} = (\mathcal{J}_B)_x$$

Then

$$\mathsf{R}_B(\mathcal{E}) = \sum_{x \in X} \mathrm{tr}\left[Q_{x,\,B}(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*(\omega_v)_x\right]$$

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For $x \in$ ... POVM on Ω

Ancillary Hilbert space $\mathcal{V}$

$$\mathsf{Q}_x : \mathcal{A} \to \mathcal{L}(\mathcal{V}) \qquad \mathsf{Q}_{x\,B} = (\mathcal{J}_B)_x$$

Then

$$\mathsf{R}_B(\mathcal{E}) = \sum_{x \in X} \mathrm{tr}\left[\mathsf{Q}_{x,\,B}(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*(\omega_v)_x\right]$$

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Fix a quantum superinstrument

$$\mathrm{R} : \mathcal{A} \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K})) ; \mathbb{C})$$

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$$\mathrm{R}_B(\mathcal{E}) = \langle v, (\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})(\mathcal{J}_B) v \rangle \quad \forall \mathcal{E} \in \mathrm{CB}(\ell^\infty(X), \mathcal{L}(\mathcal{K})), \; B \in \mathcal{A}$$

Unit vector $v \in \mathcal{K} \otimes \mathcal{V}$

For $x \in$ ... POVM on Ω

$$Q_x : \mathcal{A} \to \mathcal{L}(\mathcal{V}) \qquad Q_{x\,B} = (\mathcal{J}_B)_x$$

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For $x \in$

Quantum instrument

$$\mathcal{J} : \mathcal{A} \longrightarrow \mathrm{CP}(\mathbb{C}, \ell^\infty(X) \bar{\otimes} \mathcal{L}(\mathcal{V})) \simeq \ell^\infty(X; \mathcal{L}(\mathcal{V}))_+$$

Then

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By dilation theorem (in the Schrödinger picture)

$$\mathsf{R}_B(\mathcal{E}) = [\mathcal{J}_{B*}(\mathcal{E} \otimes \mathcal{I}_{\mathcal{V}})_*](\omega_v) \quad \forall \mathcal{E} \in \mathrm{CB}\left(\ell^\infty(X), \mathcal{L}(\mathcal{K})\right), \; B \in \mathcal{A}$$

Notation

ω_v : ortogonal projection on $\mathbb{C}v$ (rank-1 element in $\mathcal{T}(\mathcal{K} \otimes \mathcal{V})$)

Then

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Prepare a pure bipartite state ω_v in $\mathcal{K} \otimes \mathcal{V}$

An example: Quantum testers

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Measure the POVM $\mathcal{E}$ on $\mathcal{K}$, thus obtaining the outcome $x \in X$

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Conditionally on x, measure Q_x on $\mathcal{V}$, and obtain an outcome in B

Index

Summary

1. We have given a general definition of supermaps
2. We have provided a dilation theorem for deterministic and probabilistic supermaps
3. We have characterized quantum superinstruments on the space of quantum channels
4. We have shown some applications

Open problems

1. How can the dilation theorems be extended to generic supermaps (not deterministic nor probabilistic)?
2. Is there a topology on $\mathrm{CB}\left(\mathcal{M}, \mathcal{N}\right)$ such that normality of supermaps is equivalent to continuity?

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