CURRICULUM VITÆ ET STUDIORUM
of Alessandro Toigo

Anagraphical data

Surname: **Toigo**
Name: **Alessandro**
Born in: **Genoa (Italy)**
Date of birth: **4th July 1977**

Education and career

10/10/2001: I graduated in Physics at the University of Genoa with a thesis entitled *Modelli cosmologici in relatività generale* (supervisor Prof. E. Massa) with the mark of 110/110 *cum laude*.
01/02/2002 - 31/01/2005: I got a 3 years grant from the University of Genoa to attend the doctoral course in Physics (XVII cycle).
15/04/2005: I achieved the title of Ph.D. in Physics presenting the thesis entitled *Positive operator meaures, generalised imprimitivity theorem and their applications* (supervisors Profs. G. Cassinelli and P. Lahti).
01/02/2005 - 31/10/2005: I got a 9 months postdoctoral research grant issued by Fondazione Cassa di Risparmio di Genova e Imperia, aimed at continuing my research activity at the Department of Physics of the University of Genoa.
19/09/2007 - 08/10/2007: I worked with a part-time contract at the Department of Physics of the University of Genoa.
02/01/2008 - 01/01/2009: I had a 1 year research grant financed by the Department of Information Science of the University of Genoa.
02/01/2009 - 31/10/2009: My previous grant at the Department of Information Science of the University of Genoa was renewed for another 1 year.
01/11/2009 - 07/03/2012: I got a 4 years (renewable for other 4 years) temporary researcher position at the Department of Mathematics of Politecnico di Milano in the scientific sector MAT/06 - Probability and mathematical statistics.
08/03/2012 - 07/03/2015: I replaced the previous position with a new 3 years temporary researcher position of "junior" type at the Department of Mathematics of Politecnico di Milano (scientific sector: MAT/06 - Probability and mathematical statistics), since I became coordinator of the local unit of the national FIRB Project "Quantum Markov Semigroups and their empirical estimation", financed by the Italian Ministry of Education, University and Research (MIUR).
03/12/2013: I got the National Scientific Qualification (ASN) pursuant to article 16 of Law 240/2010, for the level of Associate Professor in the scientific sector 01/A4 - Mathemathical Physics.
30/12/2013: I got the National Scientific Qualification (ASN) pursuant to article 16 of Law 240/2010, for the level of Associate Professor in the scientific sector 01/A3 - Mathemathical Analysis, Probablility and Mathematical Statistics.
08/03/2015 - 15/06/2015: I had a 1 year research grant at the Department of Mathematics of Politecnico di Milano financed by my FIRB Project "Quantum

Markov Semigroups and their empirical estimation".
16/06/2015 - today: I replaced the previous position with a new 3 years temporary researcher position of "senior" type at the Department of Mathematics of Politecnico di Milano (scientific sector: MAT/06 - Probability and mathematical statistics).

Research topics

Quantum probability:
- application of techniques and methods from commutative and non-commutative harmonic analysis (Fourier analysis, Heisenberg group) to the study of sequential measurements of complementary observables in quantum mechanics;
- study of measures with values in the sets of positive operators and completely positive maps covariant with respect to representations of symmetry groups;
- symmetries and state reconstruction methods in quantum tomography;
- transformations of completely positive maps and applications in quantum information theory.

Lie supergroups:
- representation theory for Lie supergroups and study of their induction process.

Non-parametric statistics:
- applications of the theory of reproducing kernel Hilbert spaces to statistical learning;
- manifold learning;
- empirical state estimation in quantum tomography.

Publications

Articles published in international journals with referees

(1) G. Cassinelli, E. De Vito, A. Toigo, Positive operator valued measures covariant with respect to an irreducible representation, J. Math. Phys. **44** No. 10 (2003) 4768-4775
(2) G. Cassinelli, E. De Vito, A. Toigo, Positive operator valued measures covariant with respect to an Abelian group, J. Math. Phys. **45** No. 1 (2004) 418-433
(3) C. Carmeli, G. Cassinelli, E. De Vito, A. Toigo, B. Vacchini, A complete characterization of phase space measurements, J. Phys. A **37** No. 18 (2004) 5057-5066
(4) C. Carmeli, T. Heinonen, A. Toigo, Position and momentum observables on $\mathbb{R}$ and on $\mathbb{R}^3$, J. Math. Phys. **45** No. 6 (2004) 2526-2539
(5) C. Carmeli, T. Heinonen, A. Toigo, On the coexistence of position and momentum observables, J. Phys. A **38** No. 23 (2005) 5253-5266
(6) C. Carmeli, G. Cassinelli, A. Toigo, Unitary representations of super groups and Mackey theory, Bulg. J. Phys. **33** No. s2 (2006) 269-279
(7) C. Carmeli, G. Cassinelli, A. Toigo, V. S. Varadarajan, Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles, Comm. Math. Phys. **263** No. 1 (2006) 217-258
(8) C. Carmeli, E. De Vito, A. Toigo, Vector valued reproducing kernel Hilbert spaces of integrable functions and Mercer theorem, Anal. Appl. **4** No. 4 (2006) 377-408

(9) C. Carmeli, T. Heinonen, A. Toigo, Intrinsic unsharpness and approximate repeatability of quantum measurements, J. Phys. A **40** No. 6 (2007) 1303-1323
(10) C. Carmeli, T. Heinonen, A. Toigo, Why unsharp observables?, Int. J. Theor. Phys. **47** No. 1 (2008) 81-89
(11) P. Albini, A. Toigo, V. Umanità, Relations between convergence rates in Schatten p-norms, J. Math. Phys. **49** No. 1 (2008) 013504
(12) C. Carmeli, T. Heinosaari, J. P. Pellonpää, A. Toigo, Extremal covariant positive operator valued measures: the case of a compact symmetry group, J. Math. Phys. **49** No. 6 (2008) 063504
(13) C. Carmeli, T. Heinosaari, J. P. Pellonpää, A. Toigo, Optimal covariant observables: the case of a compact symmetry group and phase observables, J. Phys. A **42** No. 14 (2009) 145304
(14) S. T. Ali, C. Carmeli, T. Heinosaari, A. Toigo, Commutative POVMs and Fuzzy Observables, Found. Phys. **39** No. 6 (2009) 593-612
(15) P. Albini, E. De Vito, A. Toigo, Quantum homodyne tomography as an informationally complete positive operator valued measure, J. Phys. A **42** No. 29 (2009) 295302
(16) C. Carmeli, T. Heinosaari, A. Toigo, Covariant quantum instruments, J. Funct. Anal. **257** No. 11 (2009) 3353-3374
(17) C. Carmeli, E. De Vito, A. Toigo, V. Umanità, Vector valued reproducing kernel Hilbert spaces and universality, Anal. Appl. **8** No. 1 (2010) 19-61
(18) C. Carmeli, T. Heinosaari, A. Toigo, Sequential measurements of conjugate observables, J. Phys. A **44** No. 28 (2011) 285304
(19) C. Carmeli, G. Cassinelli, A. Toigo, V. S. Varadarajan, Erratum to: Unitary Representations of Super Lie Groups and Applications to the Classification and Multiplet Structure of Super Particles, Comm. Math. Phys. **307** No. 2 (2011) 565-566
(20) C. Carmeli, T. Heinosaari, A. Toigo, Informationally complete joint measurements on finite quantum systems, Phys. Rev. A **85** No. 1 (2012) 012109
(21) G. Chiribella, A. Toigo, V. Umanità, Normal completely positive maps on the space of quantum operations, Open Syst. Inf. Dyn. **20** No. 1 (2013) 1350003
(22) C. Carmeli, T. Heinosaari, A. Toigo, Minimal covariant observables identifying all pure states, Phys. Lett. A **377** No. 21-22 (2013) 1407-1415
(23) R. Beneduci, T. J. Bullock, P. Busch, C. Carmeli, T. Heinosaari, A. Toigo, Operational link between mutually unbiased bases and symmetric informationally complete positive operator-valued measures, Phys. Rev. A **88** No. 3 (2013) 032312
(24) C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, Tasks and premises in quantum state determination, J. Phys. A **47** No. 7 (2014) 075302
(25) E. De Vito, L. Rosasco, A. Toigo, Learning sets with separating kernels, Appl. Comput. Harmon. Anal. **37** No. 2 (2014) 185217
(26) T. Heinosaari, J. Schultz, A. Toigo, M. Ziman, Maximally incompatible quantum observables, Phys. Lett. A **378** No. 24-25 (2014) 1695-1699
(27) C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, Nonuniqueness of phase retrieval for three fractional Fourier transforms, Appl. Comput. Harmon. Anal. **39** No. 2 (2015) 339-346
(28) C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, Expanding the principle of local distinguishability, Phys. Rev. A **91** No. 4 (2015) 042121

Proceedings

(29) E. De Vito, L. Rosasco, A. Toigo, Spectral Regularization for Support Estimation, Advances in Neural Information Processing Systems 23: 24th Annual Conference on Neural Information Processing Systems 2010, NIPS 2010

(30) G. Chiribella, A. Toigo, V. Umanità, Completely positive transformations of quantum operations, QP-PQ Quantum Probability and White Noise Analysis 29: 32nd Conference on Quantum Probability and Related Topics 2011

Preprints

(31) C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, Quantum tomography meets the geometry of quantum state space (2014)

(32) C. Carmeli, T. Heinosaari, J. Schultz, A. Toigo, State reconstruction in quantum tomography with incomplete measurements (2014)

Theses

(33) A. Toigo, Modelli cosmologici in relatività generale, graduation thesis (2001)

(34) A. Toigo, Positive operator measures, generalised imprimitvity theorem and their applications, Ph.D. thesis (2005)

Workshops and conferences

Conferences and schools

(1) First national meeting "Problemi Matematici in Meccanica Quantistica", Modena, 18-20th December 2003

(2) 10th conference "Problemi Attuali di Fisica Teorica", Vietri, 2-7th April 2004

(3) 11th international conference"Symmetry Methods in Physics", Prague, 21-24th June 2004

(4) Workshop "Mathematical Methods in Quantum Mechanics", Bressanone, 21-26th February 2005

(5) 32nd international conference "Quantum Probability and Related Topics", Levico, 29th May - 4th June 2011

(6) 32nd National Conference on Harmonic Analysis, Genova, 4-8th June 2012

(7) Workshop "Quantum Markov Semigroups: Decoherence and Empirical Estimates", Genova, 26-28th June 2013

(8) Workshop "Workshop on Incompatible Quantum Measurements", Munich, 9-12th September 2013

(9) 46th international conference "Symposium on Mathematical Physics", Toruń, 15-17th June 2014

(10) 2nd workshop "Quantum Markov Semigroups: Decoherence and Empirical Estimates", Genova, 29th June - 1st July 2015

Talks

(1) "Misure covarianti a valori operatoriali in meccanica quantistica: una rassegna", presented at the meeting "Problemi Matematici in Meccanica Quantistica", Modena, 18-20th December 2003

(2) "Unitary representations of super Lie groups and super semidirect products", presented at the workshop "Mathematical Methods in Quantum Mechanics", Bressanone, 21-26th February 2005

(3) "Normal completely positive maps on the space of quantum operations", presented at the 32nd international conference "Quantum Probability and Related Topics", Levico, 29th May - 4th June 2011
(4) "Dirac equation in $1+1$ Anti-de Sitter space-time: complex structure, unitarizability and representations of $SU(1,1)$", presented at the 32nd National Conference on Harmonic Analysis, Genova, 4-8th June 2012
(5) "Normal completely positive maps on the space of quantum operations", presented at the workshop "Quantum Markov Semigroups: Decoherence and Empirical Estimates", Genova, 26-28th June 2013
(6) "On the coexistence of conjugated observables on locally compact abelian groups", presented at the workshop "Workshop on Incompatible Quantum Measurements", Munich, 9-12th September 2013
(7) "Quantum tomography for finite fields", presented at the 46th international conference "Symposium on Mathematical Physics", Toruń, 15-17th June 2014
(8) "Covariant mutually unbiased bases", presented at the 2nd workshop "Quantum Markov Semigroups: Decoherence and Empirical Estimates", Genova, 29th June - 1st July 2015

Supervision of theses and postdocs

- Supervisor of the doctoral thesis "Mathematical methods in Quantum Tomography" by Dr. Paolo Albini, 20th Doctoral Course in Physics of the University of Genoa (2008)
- Scientific supervisor of Dr. Jussi Schultz for the one-year grant "State and process empirical estimation in Quantum Tomography", issued by Politecnico di Milano and financed with my FIRB project fundings (2013/14)
- Scientific supervisor of Dr. Jussi Schultz for the one-year grant "State and process reconstruction methods in Quantum Tomography", issued by Politecnico di Milano and financed with my FIRB project fundings (2014/15)

Teaching experience

- Holder of the integrated course in Analytical and Statistical Methods for the 2nd year students in Phyiscal Engineering at Politecnico di Milano, Academic Years 2013/14 and 2014/15
- Exercise sessions of the courses in Stochastic Dynamical Models and Complements of Stochastic Processes (holder: prof. F. Fagnola) for the 4th year students of the degree course in Mathematical Engineering at Politecnico di Milano, Academic Years 2009/10, 2010/11, 2011/12, 2012/13 and 2013/14
- Exercise sessions of the course in Probability and Statistics (holder: prof. A. Barchielli) for the 2nd year students of the degree course in Computer Engineering at Politecnico di Milano, Academic Years 2010/11, 2011/12 and 2012/13
- Exercise sessions of the course in Probability and Statistics (holder: prof. F. Fagnola) for the 1st year students of the degree course in Management Engineering at Politecnico di Milano, Academic Year 2010/11

Presently, my research activity is concerned with some problems in non-commutative probability and non-parametric statistics, whose solution requires techniques from functional analysis, operator theory and non-commutative harmonic analysis. Moreover, I am also involved in several aspects of quantum tomography, mainly in the development of state reconstruction techniques and their statistical analysis. Finally, in the past I also worked on representation theory of Lie supergroups. My research can be grouped in four main topics:

(1) measures with values in the sets of positive operators and completely positive maps, and their applications in quantum mechanics;
(2) quantum tomography: state reconstruction from incomplete measurements and empirical state estimation;
(3) transformations of completely positive maps;
(4) reproducing kernel Hilbert spaces and their applications in statistical learning theory and manifold learning;
(5) supersymmetries and representation theory of super Lie groups.

In the following, I provide a separate and more detailed description of each of them.

Covariant positive operator valued measures and instruments. *Positive operator valued measures* (POVMs) and measures with values in the set of completely positive maps (briefly called *instruments* in quantum physics) constitute the mathematical objects that describe the measurement process in quantum mechanics.

Indeed, the statistics of the outcomes of a measurement is completely determined by a POVM associated with the measurement itself: knowing the state of the quantum system to be measured, the POVM provides the probability distribution of the obtained results.

The notion of instrument extends the definition of POVM, and completely encodes the action of a measurement on the measured quantum system: indeed, as well as yielding the outcome statistics, an instrument also assigns the probability distribution of the final (*a posteriori*) state of the system after the measurement.

It is relevant in physics to classify the POVMs and instruments that transform in a natural (that is, *covariant*) way under the action of the symmetry groups of the quantum system under consideration. For example, POVMs or instruments that are covariant with respect to the group of spacial translations are associated to measurements of the localization of the system; similarly, POVMs or instruments which are covariant with respect to the group of translations in phase-space describe joint measurements of its position and velocity.

My research work just consisted in applying group theory and non-commutative harmonic analysis in order to determine the structure of POVMs and instruments that are covariant under the action of symmetry groups, and then studying their general properties. Concerning POVMs, we found the complete solution of this problem in three cases of particular interest:

(1) covariance with respect to an arbitrary action of an abelian group G for POVMs that are based on transitive G-spaces [2];
(2) covariance with respect to an arbitrary of a compact group G for POVMs that are based on transitive G-spaces [12];
(3) covariance with respect to an irreducible action of an arbitrary group G for POVMs that are based on transitive G-spaces with compact stability subgroup [1, 3].

Systems that transform under the action of the Galilei group are one of the most important applications in non-relativistic quantum mechanics. In this case, the results in item (3) above lead to the characterization and study of the properties of the most general joint measurement of position and momentum that can be realized on a quantum particle [3, 4, 5, 9, 10]. In particular, in [5] a relation which is analogous to Heisenberg uncertainty principle is extended to all joint position and momentum observables, and some consequences of this fact are explained.

In [26], employing some properties of invariant means on non-compact abelian groups, we proved that the position-momentum pair observables for a quantum particle and number-phase for the photon are maximally incompatible. Here we recall that the *incompatibility degree* of two observables quantifies the minimal amount of noise that one needs to add to the observables in order to make them jointly measurable. Position-momentum and number-phase observables thus have the highest incompatibility degree among all the pairs of observables in quantum mechanics. Incidentally, in [20, 26] we also proved that only infinite-dimensional quantum systems admit maximally incompatible pairs of observables: on the contrary, in finite-dimensional systems all pairs of observables share a non-trivial compatibility region, although it has not yet been possible to determine its exact shape.

When studying the set of measurements that can be performed on a quantum system, it is of remarkable interest to determine those that are optimal with respect to some assigned cost function. Mathematically speaking, this amounts to seek for the extremal points of the convex set of the associated POVMs. In item (2), we proved that there exists a one-to-one correspondence from the set of covariant POVMs and a particular class of reproducing kernel Hilbert spaces, which allows to translate extremality into a simpler algebraic property, which is much more easy to test [12]. In [13], we applied these results to quantum optics in order to determine the phase measurements of the photon which are optimal with respect to several criteria.

A particularly important case of covariant POVM, which however does not enter in the three cases listed above, is the POVM associated to the measurement of the quadrature observables in homodyne quantum tomography. We introduced and described such POVM in [15]. In particular, we studied some properties of its associated probability distrbution and its relation with the Wigner transform of the state, obtaining the reconstruction formula of the Wigner transform from the quadrature probability densities in a mathematically consistent way. Thus, we were able to prove that the reconstruction formula is valid only imposing rather restrictive regularity conditions on the state; it is relevant noticing that it can not be extended to any larger class of states in the sense of distributions.

In the framework of covariant instruments, we obtained an extension theorem that, in analogy with a similar known fact for covariant POVMs (*Mackey-Cattaneo theorem*), leads their classification problem back to the characterization of the invariant subspaces of suitable induced representations. We then used this general result to determine the form of the instruments that are covariant with respect to irreducible representations of arbitrary groups, for the purpose of applications to quantum optics and joint position and momentum measurements also in this case [16, 18].

In [14] we proved a classification theorem for commutative POVMs that is the very analogue of the usual spectral theorem for selfadjoint operators. The essential difference relies in the fact that the spectral map associated to commutative POVMs is not based on $\mathbb{R}$, as it happens in the standard case, but instead is defined on a set of probability measures. Our result is comparable with different approaches of other authors (see in particular [S. T. Ali, Lecture Notes in Mathematics vol. **905** pp. 207-228 (1982)] e [A. Jenčová,

S. Pulmannová, Rep. Math. Phys. **59** pp. 257-266 (2007)]). Moreover, restricting to the case of covariant commutative POVMs, it is interesting to analyze some relations between our classification theorem in [14] and the general dilation theorem for covariant POVMs (Mackey-Cattaneo theorem).

The previous work was done in collaboration with

(i) Prof. G. Cassinelli and Drs. P. Albini, C. Carmeli and E. De Vito, from the University of Genoa;
(ii) Dr. B. Vacchini, from the University of Milan;
(iii) Prof. P. Lahti and Drs. T. Heinosaari, J. P. Pellonpää and J. Schultz, from the University of Turku (Finland);
(iv) Prof. S. T. Ali, from the University of Concordia (Canada);
(v) Dr. M. Ziman, from the Slovak Academy of Sciences (Slovakia).

Covariant POVMs were the subject of my Ph.D. thesis [34].

Quantum tomography with incomplete measurements. Quantum tomography is the study of the methods for determining and reconstructing an unknown quantum state from a series of suitable measurements. For this purpose, one needs to measure a set of observables which are capable to distinguish *any* state from *all* other ones. A collection of observables having this property is called *informationally complete*, and it is clear that, for d-dimensional quantum systems ($d < \infty$), it needs to be made of at least $d^2 - 1$ different osservables. In other words, without having any information on the state, the number of required observables increases quadratically with the dimension.

However, a new approach to quantum tomography that has been developed in the recent years is exploring the possibility of reconstructing a quantum state from a set of incomplete measurements. The key observation at the root of this approach is the fact that, when suitable *a priori* informations about the state to be determined are known, it is possible to drastically reduce the number of measurements needed to identify it. For example, it has been recently proved that, if the system is in a pure state, only $4d-5$ observables are needed to determine it, thus showing a linear rather than quadratic increase with respect to d.

In order to generalize this result, in some recent papers we characterized under what conditions a set of observables is capable to distinguish different states when *a priori* informations on the rank of the state are known, that is, when one has some knowledge of its degree of purity [22, 24]. In particular, we proved that the effectiveness of the observables to distinguish states with *a priori* fixed rank (*low rank informational completeness*) is in direct connection with certain properties of the operator system the observables generate.

Presently, in collaboration with Dr. C. Carmeli, from the University of Genoa, and Drs. T. Heinosaari and J. Schultz, from the University of Turku, I am involved in the following applications of the characterization of low rank informational complete observables we gave in [24]:

(1) *Locality of informational completeness with* a priori *informations.* Informational completeness is a local property: this means that if two sets of observables A_1 and A_2 are capable to determine all states of the quantum systems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, their product $\mathsf{A}_1 \otimes \mathsf{A}_2$ then distinguishes all states of the composite system $\mathcal{S}_1 \otimes \mathcal{S}_2$. This seemingly surprising fact indeed has a very simple proof. However, it is not at all clear whether an analogous property still holds even in the case of observables A_1 and A_2 which are informationally complete only with respect to pure states. Indeed, in this case the problem is much more complex. Our purpose in [28] is to solve it exploiting the characterization of low rank informationally complete observables provided in [24].

(2) *Informational completeness with* a priori *informations and geometry of the quantum state space.* The effectiveness of a set of observales to distinguish states with *a priori* known features is directly related to the geometry of the convex set of states and to the relative displacement of the operator system generated by the observables. We are presently studying this connection in [31] in order to solve the two following different types of problems:
   - given a set of observables, characterize the states that they can unambiguously discriminate;
   - given an *a priori* information on the unknown state (e.g., its degree of purity), determine what observables can uniquely resolve it.

(3) *Pure state informational completeness and homodyne quantum tomography.* In homodyne quantum tomography it is known that, in order to determine an arbitrary quantum state, about which no *a priori* information is known, one needs to measure an infinite set of quadratures. A natural question is whether such degeneracy is still present even when the unknown state is pure, or if in this case a finite number of quadratures is sufficient. A partial answer to this problem is contained in [27], where we proved that, if the unknown state is pure, in general only three quadratures do not suffice to determine it.

(4) *State reconstruction from incomplete measurements.* For a finite-dimensional quantum system, the problem of reconstructing an unknown state from the measurement outcomes of an informationally complete observable is a well-posed problem. In other words, although only an empirical and noisy version of the probability distribution of each observable is accessible to the experimenter – as a consequence that only a finite number of measures are at his disposal, from which he can only infer empirical approximations of the probability distributions –, however the reconstructed state is stable under the noise of the measurements: small perturbations in the statistics of the results generate small changes of the reconstructed state. But the same fact does not hold for observables that are informationally complete only under *a priori* conditions: indeed, in this case state reconstruction is an ill-posed inverse problem, and suitable regularization techniques are needed to turn it into an achievable task. We are trying in [32] to develop such techniques, study their consistency properties and, if possible, determine their convergence rates.

Empirical state estimation in quantum tomography. The most widely known example of state reconstruction in quantum tomography is the reconstruction of a single radiation mode of the electromagnetic field by means of the so called *homodyne tomography.* In this case, the knowledge of the probability distributions of quadratures (which are measurable by using homodyne detectors) allows to reconstruct the state of the system through inverse Radon transform by means of a method originally described by Vogel and Risken [K. Vogel and H. Risken, Phys. Rev. A **40** p. 2847 (1989)] and Leonhardt [U. Leonhardt, Phys. Rev. Lett. **74** p. 4101 (1995)]. D'Ariano *& coll.* later modified and extended this method to quantum systems endowed with particular symmetry groups [G. M. D'Ariano, M. G. A. Paris and M. F. Sacchi, Advances in Imaging and Electron Physics, **128** pp. 205-308 (2003)].

However, two main problems arise in this approach to quantum tomography, which up to now have been only partially addressed to in the wide literature on the topic, and mainly thanks to Gill *& coll.* [O. E. Bardoff-Nielsen, R. D. Gill and P. E. Jupp, J. Roy. Stat. Soc. (B) **65** pp. 775-816 (2003), L. M. Artiles, R. D. Gill and M. I. Guţă, J. Roy. Stat. Soc. (B) **67** pp. 109-134 (2005), L. Artiles, C. Butucea and M. I. Guţă, Ann. Statist. **35** pp. 465494 (2007)]). The first drawback is the fact that, if the state space of the system has infinite dimension, the tomographical reconstruction algorithm requires the knowledge of the probability distrbutions associated to an infinite number of observables – e.g., in homodyne tomography quadratures are an uncountable set of observables, parametrized by the relative phase of the local oscillator. Clearly, measuring an infinite set of observables is a task which is not achievable in practice. Secondly, it is not even possible to measure with absolute precision the probability distrbutions of each single observable, as one can only infer an approximate estimate of them from the laboratory data, affected by an error which in turn depends on the unknown state of the system.

My research activity consisted in the study and development of this kind of problems in quantum tomography. Indeed, the purpose is to apply some results from mathematical statistics and learning theory to find empirical state reconstruction algorithms, and prove their convergence and analyze the effectiveness of their approximation.

A first step in this direction is the development of the methods described by Gill *& coll.*, whose drawback is that they presently allow only a pointwise estimate of the Wigner function associated to the unknown state. In the Ph.D. thesis [P. Albini, Mathematical methods in Quantum Tomography, supervisors Drs. E. De Vito and A. Toigo, University of Genoa (2008)], we studied convergence in probability with respect to the L^2-norm of the empirical Wigner function obtained with the algorithm of Gill *& coll.*. Although this is only a preliminary result, as it does not yet allow any estimation of the state, anyway convergence in the L^2-norm is enough to prove convergence of the statistics of a large class of observables of the system. The next step then is to prove convergence of the empirical state with respect to the natural topology of the state space, that is, in the trace class norm. This presently seems a problem unrelated to the convergence of the Wigner function in the L^2-norm. However, some results from general opertor theory [B. Simon, *Trace ideals and their applications*, Cambridge University Press (1979)] show the possibility of some connection between the two types of convergence, since in particular they establish their equivalence in the state space. In [11] we proved a rather general result, which yields an estimate of the convergence rate of the trace class norm with respect to the Hilbert-Schmidt norm (the equivalent of the L^2-norm for Wigner functions), under the hypothesis that the eigenvalues of the unknown state decrease rapidly enough. However, at the moment the main problem is still the fact that the Wigner function reconstructed with the standard tomographical algorithms is *not* associated to any state, and in general does not even correspond to a trace class operator. For this reason, at the same time we are exploring an alternative approach, which does not involve Wigner functions, and rather makes use of the POVM associated to homodyne tomography. Partial results in this direction are contained in our already cited work [15].

In this topic, my research activity was carried out in collaboration with Drs. P. Albini, E. De Vito and V. Umanità, from the University of Genoa.

Transformations of completely positive maps. Quite recently, I began to study a particular class of superoperators – that is, maps between quantum channels acting on possibly different qantum states – which have a considerable relevance in quantum computation. Such class of superoperators (*supermaps*, using the terminology of [G. Chiribella, G. M. D'Ariano, P. Perinotti, Europhys. Lett. **83** 30004 (2008)], [G. Chiribella, G. M. D'Ariano, P. Perinotti, Phys. Rev. A **80** 022339 (2009)]) both appears in the theoretical description of programmable quantum circuits and networks, where one is interested in transforming the set of quantum channels acting on the state space of the system, and in quantum process tomography, where the aim is to measure a given but unknown quantum operation. As it happens for quantum operations, quantum physics requires that supermaps share themselves a suitable property of complete positivity. Exploiting this property, D'Ariano *& coll.* were able to prove a Stinespring-like dilation theorem for supermaps acting on quantum operations in the case of finite dimensional systems. Their result shows that every such supermap transforms the input channels by amplifying them in a larger system, letting them interact with its additional degrees of freedom, and then compressing them in the original space.

In [21, 30], we extended this result to the infinite-dimensional case, by introducing a notion of *normality* for supermaps analogous to the standard one for quantum operations, and proving that the dilation theorem of D'Ariano *& coll.* still holds for normal completely positive supermaps acting on infinite-dimenional systems. Moreover, we proved this fact in the more general setting of supermaps acting on quantum channels from quantum systems to general (possibly classical) systems, i.e. from full operator algebras to arbitrary von Neumann algebras. Finally, we gave some applications of our result, and in particular we described:

(1) the most general transformation of a set of quantum measurements (i.e. the whole class of the POVMs of the system) into quantum states, with the aim of characterizing quantum circuits;
(2) the structure of quantum testers, i.e. measurements performed on the whole set of quantum channels (in analogy with POVMs, which similarly describe an arbitrary measurement on the state space of the system).

This work originated from a suggestion of Dr. G. Chiribella, from the Perimeter Institute for Theoretical Physics (Canada), and continued in collaboration with Drs. Chiribella and V. Umanità, from the University of Genoa.

Reproducing kernel Hilbert spaces and their applications in statistical learning theory. *Reproducing kernel Hilbert spaces* (RKHSs) are mathematical objects which have a particular relevance in statistical learning, as they provide the hypothesis space of the theory in a very natural way. Similarly, RKHSs are at the root of the most regularization algorithms in mathematical statistics and inverse problems.

My skills are mainly related to the general mathematical theory of RKHSs. In a past work on this topic [8], we characterized the integrability and continuity properties of the functions in a RKHS $\mathcal{H}_K$ in terms of the corresponding properties of the associated reproducing kernel K, and in addition we studied the compactness of the embedding of $\mathcal{H}_K$ in L^p and $\mathcal{C}$ (the latter being the set of the continuos functions). We furthermore provided a generalizaton of the classical Mercer theorem to integral operators on non-compact spaces and with non-discrete spectrum. The relevance of the first result relies on the fact that to each integrable reproducing kernel one can associate a frame, and therefore the integrability of the kernel implies the existence of a reconstruction algorithm for the functions in the RKHS. On the other hand, the compactness property of the embedding in the L^p and $\mathcal{C}$ spaces applies in the analysis of many different approximation methods. Finally, our generalization of Mercer theorem yields an explicit construction of the RKHS $\mathcal{H}_K$ associated to a reproducing kernel K in several cases of interest in statistical learning, and in this way it allows to extend some standard results to the case in which the intergral operator with kernel K has continuous spectrum (e.g., this is the case of the gaussian kernel $K(x, y) = e^{-(x-y)^2}$).

In [17], we studied the *universality* property of reproducing kernels, in the sense introduced by [A. Caponnetto, C. A. Micchelli, M. Pontil and Y. Ying, J. Mach. Learn. Res. **9** pp. 16151646 (2008)]. Here we recall that a reproducing kernel $K : X \times X \to \mathbb{C}$ is called *universal* if the associated RKHS is dense in the space $\mathcal{C}(X)$ of the continuous complex functions on X. In other words, when K is universal, every reasonably regular function can be arbitrarily approximated with functions in $\mathcal{H}_K$, that is to say, the RKHSs associated to universal kernels provide the largest possible hypothesis spaces. In this framework, we proved the equivalence of the universality of K and the property of $\mathcal{H}_K$ to be dense in $L^2(\mu)$ for all probability measures μ on X. Specializing our result to the case of kernels on $\mathbb{R}^n$ that are translation invariant, we gave a simple and complete description of translation invariant universal kernels on the Euclidean space.

In addition to the previous topics, I am presently studying the applicability of some general results in [Berard, Besson, Gallot, Geom. Funct. Anal. **4** 373-398 (1994)] and some well-known methods in manifold learning [von Luxburg, Belkin, Bosquet, Ann. Stat. **36** 555-586 (2008)] to the research of algorithms for identifying and extracting the significant variables in problems with many parameters. The aim essentially is to develop techniques that allow to filter the relevant degrees of freedom in a very high dimensional problem from a relatively small number of empirical data (*manifold learning*). This approach, known in the literature as *diffusion map estimation* on manifolds, raises two main questions, that are the topic of my research: its consistency, and the correct mathematical understanding of the estimated quantities.

In [29, 25], we proposed a new method for manifold learning, that, rather than aiming at estimating the metric properties of the unknown submanifold, directly identifies it as the zero locus of a suitable function F_K, which is defined over the set of all the possible configurations and can be constructed from the sampled data. The function F_K is defined by means of a reproducing kernel K having the property to 'distinguish' different configurations (*separating kernel*). In this way, we have been able to prove that the estimated submanifold converges to the 'true' one (in a suitable sense), and, in principle, it is possible to determine its convergence rate.

This research was made with Drs. C. Carmeli, E. De Vito, L. Rosasco and V. Umanità, from the University of Genoa, during my collaboration with the research group of Prof. A. Verri on the study and applications of statistical learning theory.

Supersymmetries and representations of Lie supergroups. In the framework of supersymmetries, I was involved in some aspects of the theory of representations of Lie supergroups, with applications to the study of the supersymmetric extensions of relativistic quantum mechanics.

Indeed, the setting in which mathematicians provided a consistent formalization of supersymmetries is rather far from classical group theory, so much to require a partial restatement of the concept of representation itself. In [6, 7, 19], we worked out a definition of a representation of Lie supergroups based on the equivalence between supergroups and *super Harish-Chandra pairs*, that is, pairs made of a graded Lie algebra and an usual Lie group acting on it. This definition has the advantage to agree with the purely algebraic approach that is common in physics, and to generalize to the supersymmetric case some results of classical group theory. In particular, extending the standard induction process of representations, it is possible to reduce the problem of classifying the representations of a supergroup to the same problem for smaller subgroups.

In complete analogy with the classical case, our resuts have a natural and immediate application to the classification of the representations of the supersymmetric extensions of the Poincarè group. In this case, our work unifies and extends some results which are already known in physics, formalizing them in a precise and mathematically consistent framework.

My research on supersymmetries originated from my collaboration with Prof. G. Cassinelli and Dr. C. Carmeli, from the University of Genoa, and Prof. V. S. Varadarajan, from the University of Los Angeles (UCLA).