On the coexistence of conjugate observables on locally compact Abelian groups

Alessandro Toigo

Politecnico di Milano
INFN Sezione di Milano

Munich, September 10th, 2013

In collaboration with:

Claudio Carmeli

Università di Genova (Italy)

Teiko Henosaari

University of Turku (Finland)

Index

Index

Weyl systems and conjugate observables

- G is a locally compact second countable Abelian group
- $\hat{G}$ is the dual of G
- μ and $\hat{\mu}$ are the Haar measures of G and $\hat{G}$
- $\langle \xi , x \rangle$ is the pairing of $\xi \in \hat{G}$ and $x \in G$

Definition (Weyl system)

Two unitary representations U of G and V of $\hat{G}$ in the same Hilbert space $\mathcal{H}$ form a *Weyl system* for the pair $(G, \hat{G})$ if

1. for all $x \in G$, $\xi \in \hat{G}$
$$U_x V_\xi = \overline{\langle \xi , x \rangle} V_\xi U_x ;$$
2. there exists no nontrivial subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that
$$U_x \mathcal{H}_0 \subset \mathcal{H}_0 \qquad \text{and} \qquad V_\xi \mathcal{H}_0 \subset \mathcal{H}_0 \qquad \forall x \in G, \, \xi \in \hat{G} .$$

Weyl systems and conjugate observables

- G is a locally compact second countable Abelian group
- $\hat{G}$ is the dual of G
- μ and $\hat{\mu}$ are the Haar measures of G and $\hat{G}$
- $\langle \xi , x \rangle$ is the pairing of $\xi \in \hat{G}$ and $x \in G$

Definition (Weyl system)

Two unitary representations U of G and V of $\hat{G}$ in the same Hilbert space $\mathcal{H}$ form a *Weyl system* for the pair $(G, \hat{G})$ if

1. for all $x \in G$, $\xi \in \hat{G}$
$$U_x V_\xi = \overline{\langle \xi , x \rangle} V_\xi U_x \, ;$$
2. there exists no nontrivial subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that
$$U_x \mathcal{H}_0 \subset \mathcal{H}_0 \qquad \text{and} \qquad V_\xi \mathcal{H}_0 \subset \mathcal{H}_0 \qquad \forall x \in G, \, \xi \in \hat{G} \, .$$

Definition (Conjugate observables)

Two observables $\mathsf{A} : \mathcal{B}(G) \to \mathcal{L}(\mathcal{H})$ and $\mathsf{B} : \mathcal{B}(\hat{G}) \to \mathcal{L}(\mathcal{H})$ are *conjugated* if

$$U_x \mathsf{A}(X) U_x^* = \mathsf{A}(X + x) \qquad V_\xi \mathsf{A}(X) V_\xi^* = \mathsf{A}(X)$$
$$U_x \mathsf{B}(\Xi) U_x^* = \mathsf{B}(\Xi) \qquad V_\xi \mathsf{B}(\Xi) V_\xi^* = \mathsf{B}(\Xi + \xi)$$

for all $x \in G$, $\xi \in \hat{G}$, $X \in \mathcal{B}(G)$, $\Xi \in \mathcal{B}(\hat{G})$.

By Stone-Naimark-Ambrose-Godement Theorem,

$$U_x = \int_{\hat{G}} \overline{\langle \xi \, , \, x \rangle} \, \mathrm{d}\hat{\mathsf{B}}(\xi) \qquad V_\xi = \int_G \langle \xi \, , \, x \rangle \, \mathrm{d}\hat{\mathsf{A}}(\xi) \, .$$

where $\hat{\mathsf{A}}$ and $\hat{\mathsf{B}}$ are conjugated spectral measures.

Every Weyl system is unitarily equivalent to the *canonical* one

$$\mathcal{H} = L^2(G, \mu)$$

$$U_x f = f(\cdot - x) \qquad V_\xi f = \langle \xi \, , \cdot \rangle \, f$$

$$\hat{\mathsf{A}}(X) f = \mathbb{1}_X f \qquad \hat{\mathsf{B}}(\Xi) f = (\mathcal{F}^{-1} \mathbb{1}_\Xi) * f$$

where $\mathcal{F} : L^2(G, \mu) \to L^2(\hat{G}, \hat{\mu})$ is Fourier transform and $*$ is convolution.

By Stone-Naimark-Ambrose-Godement Theorem,

$$U_x = \int_{\hat{G}} \overline{\langle \xi , x \rangle} \, \mathrm{d}\hat{\mathsf{B}}(\xi) \qquad V_\xi = \int_G \langle \xi , x \rangle \, \mathrm{d}\hat{\mathsf{A}}(\xi) \, .$$

where $\hat{\mathsf{A}}$ and $\hat{\mathsf{B}}$ are conjugated spectral measures.

Every Weyl system is unitarily equivalent to the *canonical* one

$$\mathcal{H} = L^2(G, \mu)$$

$$U_x f = f(\cdot - x) \qquad V_\xi f = \langle \xi , \cdot \rangle \, f$$

$$\hat{\mathsf{A}}(X) f = \mathbb{1}_X f \qquad \hat{\mathsf{B}}(\Xi) f = (\mathcal{F}^{-1} \mathbb{1}_\Xi) * f$$

where $\mathcal{F} : L^2(G, \mu) \to L^2(\hat{G}, \hat{\mu})$ is Fourier transform and $*$ is convolution.

Example: $G = \hat{G} = \mathbb{R}$

$$\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$$

$$U_x = \mathrm{e}^{-ixP} \qquad V_\xi = \mathrm{e}^{i\xi Q}$$

$$P = \int_{\mathbb{R}} z \,\mathrm{d}\hat{\mathsf{A}}(z) \qquad Q = \int_{\mathbb{R}} z \,\mathrm{d}\hat{\mathsf{B}}(z)$$

Example: $G = \hat{G} = \mathbb{Z}_d$

$$\mathcal{H} = \mathbb{C}^d$$

$$U_x e_k = e_{k+x} \qquad V_\xi e_k = \mathrm{e}^{2\pi i \frac{\xi k}{d}} e_k$$

$$\hat{\mathsf{A}}(k) = |e_k\rangle \langle e_k| \qquad \hat{\mathsf{B}}(k) = \mathcal{F}^* \hat{\mathsf{A}}(k) \mathcal{F} \, .$$

In particular, the observavles Â and B̂ are *mutually unbiased*

$$\mathrm{tr}\left[\hat{\mathsf{A}}(i)\hat{\mathsf{B}}(j)\right] = 1/d \qquad \forall i, j \, .$$

Structure of conjugate observables

Theorem (Carmeli, Heinosaari, T.)

If (A, B) *are conjugate observables, there exist unique probability measures* Λ *on* G *and* Γ *on* $\hat{G}$ *such that*

$$\mathsf{A}(X) =: \mathsf{A}_\Lambda(X) = \int_G \Lambda(X - x)\, d\hat{\mathsf{A}}(x) \qquad \forall X \in \mathcal{B}(G)$$

$$\mathsf{B}(\Xi) =: \mathsf{B}_\Gamma(\Xi) = \int_{\hat{G}} \Gamma(\Xi - \xi)\, d\hat{\mathsf{B}}(\xi) \qquad \forall \Xi \in \mathcal{B}(\hat{G})\,.$$

The observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ are *noisy versions* of $(\hat{\mathsf{A}}, \hat{\mathsf{B}})$:

$$\left.\begin{aligned} p_\rho^{\mathsf{A}_\Lambda}(X) &:= \mathrm{tr}\,[\rho \mathsf{A}_\Lambda(X)] = (\Lambda * p_\rho^{\hat{\mathsf{A}}})(X) \\ p_\rho^{\mathsf{B}_\Gamma}(\Xi) &:= \mathrm{tr}\,[\rho \mathsf{B}_\Gamma(\Xi)] = (\Gamma * p_\rho^{\hat{\mathsf{B}}})(\Xi) \end{aligned}\right. \qquad \forall \rho \in \mathcal{S}(\mathcal{H})\,.$$

Structure of conjugate observables

Theorem (Carmeli, Heinosaari, T.)

If (A, B) *are conjugate observables, there exist unique probability measures* Λ *on* G *and* Γ *on* $\hat{G}$ *such that*

$$\mathsf{A}(X) =: \mathsf{A}_\Lambda(X) = \int_G \Lambda(X - x)\, d\hat{\mathsf{A}}(x) \qquad \forall X \in \mathcal{B}(G)$$

$$\mathsf{B}(\Xi) =: \mathsf{B}_\Gamma(\Xi) = \int_{\hat{G}} \Gamma(\Xi - \xi)\, d\hat{\mathsf{B}}(\xi) \qquad \forall \Xi \in \mathcal{B}(\hat{G})\,.$$

The observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ are *noisy versions* of $(\hat{\mathsf{A}}, \hat{\mathsf{B}})$:

$$\left.\begin{array}{l} p_\rho^{\mathsf{A}_\Lambda}(X) := \mathrm{tr}\,[\rho \mathsf{A}_\Lambda(X)] = (\Lambda * p_\rho^{\hat{\mathsf{A}}})(X) \\ p_\rho^{\mathsf{B}_\Gamma}(\Xi) := \mathrm{tr}\,[\rho \mathsf{B}_\Gamma(\Xi)] = (\Gamma * p_\rho^{\hat{\mathsf{B}}})(\Xi) \end{array}\right. \qquad \forall \rho \in \mathcal{S}(\mathcal{H})\,.$$

Covariant phase-space observables

Definition (Covariant phase-space observables)

An observable G : $\mathcal{B}(G \times \hat{G}) \to \mathcal{L}(\mathcal{H})$ is a *covariant phase-space observable* if

$$U_x V_\xi \mathrm{G}(Z) V_\xi^* U_x^* = \mathrm{G}(Z + (x, \xi))$$

for all $Z \in \mathcal{B}(G \times \hat{G})$, $x \in G$, $\xi \in \hat{G}$.

Theorem (Holevo; Cassinelli, De Vito, T.)

If G is a covariant phase-space observable, then there exists a unique state $\tau \in \mathcal{S}(\mathcal{H})$ such that

$$\mathrm{G}(Z) = \int_Z U_x V_\xi \tau V_\xi^* U_x^* \, d\mu(x) \, d\hat{\mu}(\xi) \qquad \forall Z \in \mathcal{B}(G \times \hat{G}) .$$

Covariant phase-space observables

Definition (Covariant phase-space observables)

An observable G : $\mathcal{B}(G \times \hat{G}) \to \mathcal{L}(\mathcal{H})$ is a *covariant phase-space observable* if

$$U_x V_\xi \mathsf{G}(Z) V_\xi^* U_x^* = \mathsf{G}(Z + (x, \xi))$$

for all $Z \in \mathcal{B}(G \times \hat{G})$, $x \in G$, $\xi \in \hat{G}$.

Theorem (Holevo; Cassinelli, De Vito, T.)

If G *is a covariant phase-space observable, then there exists a unique state* $\tau \in \mathcal{S}(\mathcal{H})$ *such that*

$$\mathsf{G}(Z) = \int_Z U_x V_\xi \tau V_\xi^* U_x^* \, d\mu(x) \, d\hat{\mu}(\xi) \qquad \forall Z \in \mathcal{B}(G \times \hat{G}) \, .$$

Main theorem

Taking the margins of a covariant phase-space observable

$$\mathsf{A}(X) = \mathsf{G}(X \times \hat{G}) \qquad \mathsf{B}(\Xi) = \mathsf{G}(G \times \Xi)$$

we obtain conjugate observables (A, B).

Definition (Joint measurability)

Conjugate observables (A, B) are *jointly measurabile* if they are the margins of a single observable G on $G \times \hat{G}$.

Remark

The joint observable G is in general not unique.

Main theorem

Taking the margins of a covariant phase-space observable

$$\mathsf{A}(X) = \mathsf{G}(X \times \hat{G}) \qquad \mathsf{B}(\Xi) = \mathsf{G}(G \times \Xi)$$

we obtain conjugate observables (A, B).

Theorem (Carmeli, Heinosaari, T.)

Conjugate observables (A, B) *are jointly measurabile if and only if they are the margins of a covariant phase-space observable* G *on* $G \times \hat{G}$.

Remark

The generating state τ of the covariant joint observable $\mathsf{G} = \mathsf{G}_\tau$ is in general not unique.

Corollaries

Corollary

For conjugate observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$*, these are equivalent facts:*

- $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ *are jointly measurable;*
- *there is a state* $\tau \in \mathcal{S}(\mathcal{H})$ *such that*

$$\Lambda(X) = \mathrm{tr}\left[\tau \hat{\mathsf{A}}(-X)\right] \qquad \Gamma(\Xi) = \mathrm{tr}\left[\tau \hat{\mathsf{B}}(-\Xi)\right] \; ;$$

- *there is a vector* $\psi \in \mathcal{H} \otimes \mathcal{H}$ *such that*

$$\Lambda(X) = \left\langle \psi \, , \, (\hat{\mathsf{A}}(-X) \otimes \mathbb{1})\psi \right\rangle \qquad \Gamma(\Xi) = \left\langle \psi \, , \, (\hat{\mathsf{B}}(-X) \otimes \mathbb{1})\psi \right\rangle \, .$$

Corollary

Conjugate observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ *are jointly measurable only if* Λ *and* Γ *have densities w.r.t. the Haar measures* μ *and* $\hat{\mu}$.

Corollaries

Corollary

For conjugate observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$, *these are equivalent facts:*

- $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ *are jointly measurable;*
- *there is a state* $\tau \in \mathcal{S}(\mathcal{H})$ *such that*

$$\Lambda(X) = \mathrm{tr}\left[\tau \hat{\mathsf{A}}(-X)\right] \qquad \Gamma(\Xi) = \mathrm{tr}\left[\tau \hat{\mathsf{B}}(-\Xi)\right] ;$$

- *there is a vector* $\psi \in \mathcal{H} \otimes \mathcal{H}$ *such that*

$$\Lambda(X) = \left\langle \psi , (\hat{\mathsf{A}}(-X) \otimes \mathbb{1})\psi \right\rangle \qquad \Gamma(\Xi) = \left\langle \psi , (\hat{\mathsf{B}}(-X) \otimes \mathbb{1})\psi \right\rangle .$$

Corollary

Conjugate observables $(\mathsf{A}_\Lambda, \mathsf{B}_\Gamma)$ *are jointly measurable only if* Λ *and* Γ *have densities w.r.t. the Haar measures* μ *and* $\hat{\mu}$.

Index

Case $G = \hat{G} = \mathbb{R}$

If p is a probability measure on $\mathbb{R}$,

$$\mathrm{Var}(p) = \int_{\mathbb{R}} \left(y - \int_{\mathbb{R}} x \, \mathrm{d}p(x) \right)^2 \mathrm{d}p(y) \, .$$

Proposition

If the conjugate observables (A, B) *are jointly measurable, then for all states* $\rho \in \mathcal{S}(\mathcal{H})$

$$\mathrm{Var}(p_\rho^{\mathsf{A}}) \mathrm{Var}(p_\rho^{\mathsf{B}}) \geq 1 \, .$$

Case $G = \hat{G} = \mathbb{R}$

If p is a probability measure on $\mathbb{R}$,

$$\mathrm{Var}(p) = \int_{\mathbb{R}} \left(y - \int_{\mathbb{R}} x \, \mathrm{d}p(x) \right)^2 \mathrm{d}p(y) \, .$$

Proposition

If the conjugate observables (A, B) *are jointly measurable, then for all states* $\rho \in \mathcal{S}(\mathcal{H})$

$$\mathrm{Var}(p^{\mathsf{A}}_{\rho}) \mathrm{Var}(p^{\mathsf{B}}_{\rho}) \geq 1 \, .$$

Case $G = \hat{G} = \mathbb{Z}_d$

If $\lambda, \gamma \in [0, 1]$, we define conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$

$$\mathsf{A}_\lambda(k) = \lambda \hat{\mathsf{A}}(k) + (1 - \lambda) \frac{1}{d} \mathbb{1}$$

$$\mathsf{B}_\gamma(k) = \gamma \hat{\mathsf{B}}(k) + (1 - \gamma) \frac{1}{d} \mathbb{1} \, .$$

Proposition

For all $\lambda \in [0, 1]$, let

$$\gamma_{\max}(\lambda) = \frac{(d - 2)(1 - \lambda) + 2\sqrt{(1 - d)\lambda^2 + (d - 2)\lambda + 1}}{d} \, .$$

Then, the conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ are jointly measurable if and only if $0 \leq \gamma \leq \gamma_{\max}(\lambda)$.

Case $G = \hat{G} = \mathbb{Z}_d$

If $\lambda, \gamma \in [0, 1]$, we define conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$

$$\mathsf{A}_\lambda(k) = \lambda \hat{\mathsf{A}}(k) + (1 - \lambda)\,\frac{1}{d}\,\mathbb{1}$$

$$\mathsf{B}_\gamma(k) = \gamma \hat{\mathsf{B}}(k) + (1 - \gamma)\,\frac{1}{d}\,\mathbb{1}\,.$$

Proposition

For all $\lambda \in [0, 1]$*, let*

$$\gamma_{\max}(\lambda) = \frac{(d-2)(1-\lambda) + 2\sqrt{(1-d)\lambda^2 + (d-2)\lambda + 1}}{d}\,.$$

Then, the conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *are jointly measurable if and only if* $0 \leq \gamma \leq \gamma_{\max}(\lambda)$*.*

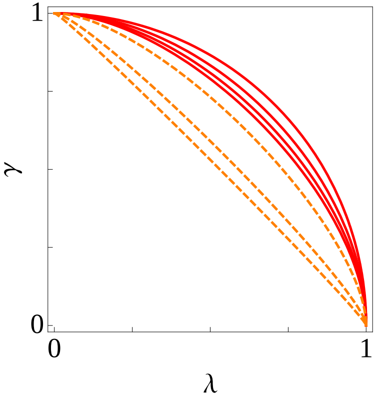

Figure: The boundary curve $\lambda \mapsto \gamma_{\text{max}}(\lambda)$ for $d = 2, 3, 4, 5$ (red solid curves) and for $d = 10, 100, 1000$ (orange dashed curves).

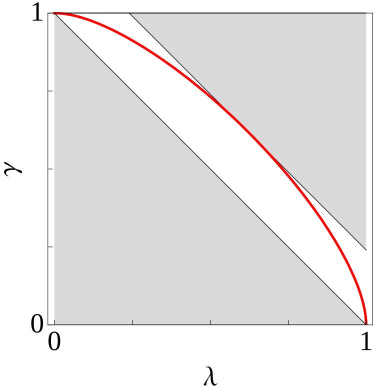

Figure: In this picture $d = 10$. The grey regions represent necessary and sufficient linear conditions.

Corollary

Conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$

- *are jointly measurable if* $\gamma + \lambda \leq 1$;
- *are not jointly measurable if* $\gamma + \lambda \geq 1 + \frac{\sqrt{d}-1}{d-1}$.

Proposition

If $\gamma = \gamma_{\max}(\lambda)$, *then the conjugate observables* $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *have a unique joint observable. This unique joint observable is the covariant phase-space observable* G_τ *with*

$$\tau = |\chi_\lambda\rangle \langle\chi_\lambda| \, , \qquad \chi_\lambda = \alpha_\lambda e_0 + \beta_\lambda \mathcal{F}^* e_0 \, ,$$

where

$$\alpha_\lambda = \frac{1}{\sqrt{d}} \left[\sqrt{(d-1)\lambda + 1} - \sqrt{1-\lambda} \right] \qquad \beta_\lambda = \sqrt{1-\lambda} \, .$$

Corollary

Conjugate observables $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$

- *are jointly measurable if* $\gamma + \lambda \leq 1$;
- *are not jointly measurable if* $\gamma + \lambda \geq 1 + \frac{\sqrt{d}-1}{d-1}$.

Proposition

If $\gamma = \gamma_{\max}(\lambda)$, *then the conjugate observables* $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *have a unique joint observable. This unique joint observable is the covariant phase-space observable* G_τ *with*

$$\tau = |\chi_\lambda\rangle \langle\chi_\lambda| \, , \qquad \chi_\lambda = \alpha_\lambda e_0 + \beta_\lambda \mathcal{F}^* e_0 \, ,$$

where

$$\alpha_\lambda = \frac{1}{\sqrt{d}} \left[\sqrt{(d-1)\lambda + 1} - \sqrt{1-\lambda} \right] \qquad \beta_\lambda = \sqrt{1-\lambda} \, .$$

If p and r are probability distributions on $\mathbb{Z}_d$, define

$$\mathsf{A}_{\lambda;p}(j) = \lambda \mathsf{A}(j) + (1-\lambda)p(j)\mathbb{1}$$
$$\mathsf{B}_{\gamma;r}(k) = \gamma \mathsf{B}(k) + (1-\gamma)r(k)\mathbb{1}\,.$$

The observables $\mathsf{A}_{\lambda;p}$ and $\mathsf{B}_{\gamma;r}$ are *not* covariant. However, the following still holds.

Proposition

The observables $\mathsf{A}_{\lambda;p}$ and $\mathsf{B}_{\gamma;r}$ are jointly measurable if and only if $0 \leq \gamma \leq \gamma_{\max}(\lambda)$.

If p and r are probability distributions on $\mathbb{Z}_d$, define

$$\mathsf{A}_{\lambda;p}(j) = \lambda \mathsf{A}(j) + (1-\lambda) p(j) \mathbb{1}$$
$$\mathsf{B}_{\gamma;r}(k) = \gamma \mathsf{B}(k) + (1-\gamma) r(k) \mathbb{1} \, .$$

The observables $\mathsf{A}_{\lambda;p}$ and $\mathsf{B}_{\gamma;r}$ are *not* covariant. However, the following still holds.

Proposition

The observables $\mathsf{A}_{\lambda;p}$ and $\mathsf{B}_{\gamma;r}$ are jointly measurable if and only if $0 \leq \gamma \leq \gamma_{\max}(\lambda)$.

Sequential implementation

Define the A_λ-compatible instrument

$$\mathcal{I} : \mathbb{Z}_d \times \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$$
$$\mathcal{I}(k, \rho) = \sqrt{\mathsf{A}_\lambda(k)}\rho\sqrt{\mathsf{A}_\lambda(k)}$$

Then, $\mathcal{I}$ is a *generalized position instrument*

$$U_i V_j \mathcal{I}(k, V_j^* U_i^* \rho U_i V_j) V_j^* U_i^* = \mathcal{I}(i + k, \rho)$$

and a sequential measurement of $\mathcal{I}$ followed by B gives the covariant phase-space observable

$$\mathsf{G}(i, j) = \mathcal{I}_i^*(\mathsf{B}(j)) = \sqrt{\mathsf{A}_\lambda(i)}\mathsf{B}_\gamma(j)\sqrt{\mathsf{A}_\lambda(i)} .$$

It is easy to check that $\mathsf{G} \equiv \mathsf{G}_\tau$, with $\tau = |\chi_\lambda\rangle\langle\chi_\lambda|$. Therefore, G is a joint observable of $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ with $\gamma \equiv \gamma_{\max}(\lambda)$.

Sequential implementation

Define the A_λ-compatible instrument

$$\mathcal{I} : \mathbb{Z}_d \times \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$$
$$\mathcal{I}(k, \rho) = \sqrt{A_\lambda(k)}\rho\sqrt{A_\lambda(k)}$$

Then, $\mathcal{I}$ is a *generalized position instrument*

$$U_i V_j \mathcal{I}(k, V_j^* U_i^* \rho U_i V_j) V_j^* U_i^* = \mathcal{I}(i + k, \rho)$$

and a sequential measurement of $\mathcal{I}$ followed by B gives the covariant phase-space observable

$$G(i, j) = \mathcal{I}_i^*(B(j)) = \sqrt{A_\lambda(i)} B_\gamma(j) \sqrt{A_\lambda(i)} .$$

It is easy to check that $G \equiv G_\tau$, with $\tau = |\chi_\lambda\rangle\langle\chi_\lambda|$. Therefore, G is a joint observable of (A_λ, B_γ) with $\gamma \equiv \gamma_{\max}(\lambda)$.

Sequential implementation

Define the A_λ-compatible instrument

$$\mathcal{I} : \mathbb{Z}_d \times \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$$
$$\mathcal{I}(k, \rho) = \sqrt{\mathsf{A}_\lambda(k)}\rho\sqrt{\mathsf{A}_\lambda(k)}$$

Then, $\mathcal{I}$ is a *generalized position instrument*

$$U_i V_j \mathcal{I}(k, V_j^* U_i^* \rho U_i V_j) V_j^* U_i^* = \mathcal{I}(i + k, \rho)$$

and a sequential measurement of $\mathcal{I}$ followed by B gives the covariant phase-space observable

$$\mathsf{G}(i, j) = \mathcal{I}_i^*(\mathsf{B}(j)) = \sqrt{\mathsf{A}_\lambda(i)}\mathsf{B}_\gamma(j)\sqrt{\mathsf{A}_\lambda(i)}\,.$$

It is easy to check that $\mathsf{G} \equiv \mathsf{G}_\tau$, with $\tau = |\chi_\lambda\rangle\langle\chi_\lambda|$. Therefore, G is a joint observable of $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ with $\gamma \equiv \gamma_{\max}(\lambda)$.

Informational completeness

Proposition

Suppose $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *are conjugate observables.*

- *If* $\lambda \notin \{0, 1\}$ *and* $0 < \gamma < \gamma_{\max}(\lambda)$*, then they have an informationally complete joint observable.*
- *If* $\lambda \notin \{0, 1\}$ *and* $\gamma = \gamma_{\max}(\lambda)$*, then their unique joint observable is informationally complete if and only if d is odd.*
- *If* $\lambda = 0$ *or* $\gamma = 0$*, then they have no informationally complete joint observable.*

Informational completeness

Proposition

Suppose $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *are conjugate observables.*

- *If* $\lambda \notin \{0, 1\}$ *and* $0 < \gamma < \gamma_{\max}(\lambda)$*, then they have an informationally complete joint observable.*
- *If* $\lambda \notin \{0, 1\}$ *and* $\gamma = \gamma_{\max}(\lambda)$*, then their unique joint observable is informationally complete if and only if* d *is odd.*
- *If* $\lambda = 0$ *or* $\gamma = 0$*, then they have no informationally complete joint observable.*

Informational completeness

Proposition

Suppose $(\mathsf{A}_\lambda, \mathsf{B}_\gamma)$ *are conjugate observables.*

- *If* $\lambda \notin \{0, 1\}$ *and* $0 < \gamma < \gamma_{\max}(\lambda)$, *then they have an informationally complete joint observable.*
- *If* $\lambda \notin \{0, 1\}$ *and* $\gamma = \gamma_{\max}(\lambda)$, *then their unique joint observable is informationally complete if and only if* d *is odd.*
- *If* $\lambda = 0$ *or* $\gamma = 0$, *then they have no informationally complete joint observable.*

Index

Case $|G| = |\hat{G}| < \infty$.

If conjugate observables (A, B) are jointly measurable, with joint observable G on $G \times \hat{G}$, we define

$$\tilde{\mathsf{G}}(x, \xi) = \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} \underbrace{U_y^* V_\eta^* \mathsf{G}(x + y, \xi + \eta) V_\eta U_y}_{\mathsf{G}_{(y,\eta)}(x,\xi)} .$$

Then, $\tilde{\mathsf{G}}$ is a covariant phase-space observable with margins

$$\tilde{\mathsf{G}}(\{x\} \times \hat{G}) = \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} U_y^* V_\eta^* \sum_{\xi \in \hat{G}} \mathsf{G}(x + y, \xi + \eta) V_\eta U_y$$
$$= \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} U_y^* V_\eta^* \mathsf{A}(x + y) V_\eta U_y = \mathsf{A}(x) .$$

Case $|G| = |\hat{G}| < \infty$.

If conjugate observables (A, B) are jointly measurable, with joint observable G on $G \times \hat{G}$, we define

$$\tilde{\mathsf{G}}(x, \xi) = \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} \underbrace{U_y^* V_\eta^* \mathsf{G}(x + y, \xi + \eta) V_\eta U_y}_{\mathsf{G}_{(y,\eta)}(x,\xi)} .$$

Then, $\tilde{\mathsf{G}}$ is a covariant phase-space observable with margins

$$\tilde{\mathsf{G}}(\{x\} \times \hat{G}) = \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} U_y^* V_\eta^* \sum_{\xi \in \hat{G}} \mathsf{G}(x + y, \xi + \eta) V_\eta U_y$$

$$= \frac{1}{|G|^2} \sum_{(y,\eta) \in G \times \hat{G}} U_y^* V_\eta^* \mathsf{A}(x + y) V_\eta U_y = \mathsf{A}(x) .$$

Case $|G| = |\hat{G}| < \infty$.

If conjugate observables (A, B) are jointly measurable, with joint observable G on $G \times \hat{G}$, we define

$$\tilde{\mathsf{G}}(x,\xi) = \frac{1}{|G|^2} \sum_{(y,\eta)\in G\times\hat{G}} \underbrace{U_y^* V_\eta^* \mathsf{G}(x+y,\xi+\eta) V_\eta U_y}_{\mathsf{G}_{(y,\eta)}(x,\xi)} .$$

Then, $\tilde{\mathsf{G}}$ is a covariant phase-space observable with margins

$$\tilde{\mathsf{G}}(G\times\{\xi\}) = \frac{1}{|G|^2} \sum_{(y,\eta)\in G\times\hat{G}} U_y^* V_\eta^* \sum_{x\in G} \mathsf{G}(x+y,\xi+\eta) V_\eta U_y$$

$$= \frac{1}{|G|^2} \sum_{(y,\eta)\in G\times\hat{G}} U_y^* V_\eta^* \mathsf{B}(\xi+\eta) V_\eta U_y = \mathsf{B}(\xi)$$

Case $|G| = |\hat{G}| = \infty$

Definition (Invariant mean)

Let Ω be a locally compact separable metric space. An *operator valued mean* on Ω is a linear map

$$\mathrm{M} : BC(\Omega) \to \mathcal{L}(\mathcal{K})$$

such that

(i) $\mathrm{M}(f) \geq 0$ if $f \geq 0$;

(ii) $\mathrm{M}(1) = \mathbb{1}$.

An observable M on Ω defines a operator valued mean on Ω

$$\mathrm{M}(f) := \int_\Omega f(\omega)\, \mathrm{dM}(\omega) \qquad \forall f \in BC(\Omega)\,.$$

Case $|G| = |\hat{G}| = \infty$

Definition (Invariant mean)

Let Ω be a locally compact separable metric space. An *operator valued mean* on Ω is a linear map

$$\mathsf{M} : BC(\Omega) \to \mathcal{L}(\mathcal{K})$$

such that

(i) $\mathsf{M}(f) \geq 0$ if $f \geq 0$;

(ii) $\mathsf{M}(1) = \mathbb{1}$.

An observable M on Ω defines a operator valued mean on Ω

Remark

If $\mathcal{K} = \mathbb{C}$, then M is a *mean*, that is, a normalized positive linear functional on $BC(\Omega)$.

Case $|G| = |\hat{G}| = \infty$

Definition (Invariant mean)

Let Ω be a locally compact separable metric space. An *operator valued mean* on Ω is a linear map

$$\mathrm{M} : BC(\Omega) \to \mathcal{L}(\mathcal{K})$$

such that

(i) $\mathrm{M}(f) \geq 0$ if $f \geq 0$;

(ii) $\mathrm{M}(1) = \mathbb{1}$.

An observable M on Ω defines a operator valued mean on Ω

$$\mathrm{M}(f) := \int_\Omega f(\omega)\, \mathrm{dM}(\omega) \qquad \forall f \in BC(\Omega)\,.$$

If M is an operator valued mean on Ω, define

$$\mathsf{M}(\infty) = \mathbb{1} - \mathrm{LUB}\{\mathsf{M}(f) \mid f \in C_c(\Omega),\ 0 \leq f \leq 1\}.$$

There exists a unique operator valued measure M_0 on Ω such that

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in C_c(\Omega).$$

For such M_0,

$$\mathsf{M}_0(\Omega) = \mathbb{1} - \mathsf{M}(\infty).$$

Proposition

If $\mathsf{M}(\infty) = 0$*, then*

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in BC(\Omega).$$

If M is an operator valued mean on Ω, define

$$\mathsf{M}(\infty) = \mathbb{1} - \mathrm{LUB}\{\mathsf{M}(f) \mid f \in C_c(\Omega),\ 0 \leq f \leq 1\}.$$

There exists a unique operator valued measure M_0 on Ω such that

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in C_c(\Omega).$$

For such M_0,

$$\mathsf{M}_0(\Omega) = \mathbb{1} - \mathsf{M}(\infty).$$

Proposition

If $\mathsf{M}(\infty) = 0$*, then*

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in BC(\Omega).$$

If M is an operator valued mean on Ω, define

$$\mathsf{M}(\infty) = \mathbb{1} - \mathrm{LUB}\{\mathsf{M}(f) \mid f \in C_c(\Omega),\ 0 \leq f \leq 1\}\,.$$

There exists a unique operator valued measure M_0 on Ω such that

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in C_c(\Omega)\,.$$

For such M_0,

$$\mathsf{M}_0(\Omega) = \mathbb{1} - \mathsf{M}(\infty)\,.$$

Proposition

If $\mathsf{M}(\infty) = 0$*, then*

$$\mathsf{M}(f) = \mathsf{M}_0(f) \qquad \forall f \in BC(\Omega)\,.$$

Suppose $\Omega = \Omega_1 \times \Omega_2$.
For $f_i \in BC(\Omega_i)$, define $\widetilde{f}_i \in BC(\Omega)$ as

$$\widetilde{f}_i(\omega_1, \omega_2) = f_i(\omega_i) \, .$$

Definition (Margins of an operator valued mean)

If M is an operator valued mean on Ω, then the *i-th margin* of M is the operator valued mean M_i on Ω_i given by

$$\mathrm{M}_i(f_i) = \mathrm{M}(\widetilde{f}_i) \qquad \forall f_i \in BC(\Omega_i) \, .$$

Proposition

Let M *be an operator valued mean on* Ω.

(i) *If* $\mathrm{M}_1(\infty) = \mathrm{M}_2(\infty) = 0$, *then* $\mathrm{M}(\infty) = 0$.

(ii) *If* $\mathrm{M}(\infty) = 0$, *then* $(\mathrm{M}_0)_i = (\mathrm{M}_i)_0$.

Suppose $\Omega = \Omega_1 \times \Omega_2$.
For $f_i \in BC(\Omega_i)$, define $\widetilde{f_i} \in BC(\Omega)$ as

$$\widetilde{f_i}(\omega_1, \omega_2) = f_i(\omega_i) .$$

Definition (Margins of an operator valued mean)

If M is an operator valued mean on Ω, then the *i-th margin* of M is the operator valued mean M_i on Ω_i given by

$$\mathrm{M}_i(f_i) = \mathrm{M}(\widetilde{f_i}) \qquad \forall f_i \in BC(\Omega_i) .$$

Proposition

Let M *be an operator valued mean on* Ω*.*

(i) *If* $\mathrm{M}_1(\infty) = \mathrm{M}_2(\infty) = 0$*, then* $\mathrm{M}(\infty) = 0$*.*

(ii) *If* $\mathrm{M}(\infty) = 0$*, then* $(\mathrm{M}_0)_i = (\mathrm{M}_i)_0$*.*

For all $f \in BC(G \times \hat{G})$ and $(x, \xi) \in G \times \hat{G}$, let

$$f_{(x,\xi)} = f(\cdot + x, \cdot + \xi) .$$

Definition (Invariant mean)

A mean m on $G \times \hat{G}$ is *invariant* if

$$m(f_{(x,\xi)}) = m(f) \qquad \forall f \in BC(G \times \hat{G}), (x, \xi) \in G \times \hat{G} .$$

Theorem (E. Hewitt, K. A. Ross, Th. IV.17.5)

There exists an invariant mean on $G \times \hat{G}$.

For all $f \in BC(G \times \hat{G})$ and $(x, \xi) \in G \times \hat{G}$, let

$$f_{(x,\xi)} = f(\cdot + x, \cdot + \xi) .$$

Definition (Invariant mean)

A mean m on $G \times \hat{G}$ is *invariant* if

$$m(f_{(x,\xi)}) = m(f) \qquad \forall f \in BC(G \times \hat{G}),\ (x, \xi) \in G \times \hat{G} .$$

Theorem (E. Hewitt, K. A. Ross, Th. IV.17.5)

There exists an invariant mean on $G \times \hat{G}$.

Let M be a joint observable of A_Λ and B_Γ.

For all $\varphi, \psi \in \mathcal{H}$ and $(x, \xi) \in G \times \hat{G}$, define

$$\Theta[f; \varphi, \psi](x, \xi) = \left\langle \mathsf{M}(f_{(x,\xi)}) V_\xi^* U_x^* \varphi \, , \, V_\xi^* U_x^* \psi \right\rangle .$$

Then, $\Theta[f; \varphi, \psi] \in BC(G \times \hat{G})$, and

$$\Theta[f_{(x,\xi)}; \varphi, \psi] = \Theta[f; U_x V_\xi \varphi, U_x V_\xi \psi]_{(x,\xi)}$$

$$\Theta[\widetilde{f_1}; \varphi, \psi](x, \xi) = \langle \mathsf{A}_\Lambda(f) \varphi \, , \, \psi \rangle \qquad \forall f \in BC(G)$$

$$\Theta[\widetilde{g_2}; \varphi, \psi](x, \xi) = \langle \mathsf{B}_\Gamma(g) \varphi \, , \, \psi \rangle \qquad \forall g \in BC(\hat{G}) .$$

Let M be a joint observable of A_Λ and B_Γ.
For all $\varphi, \psi \in \mathcal{H}$ and $(x, \xi) \in G \times \hat{G}$, define

$$\Theta[f; \varphi, \psi](x, \xi) = \left\langle \mathsf{M}(f_{(x,\xi)}) V_\xi^* U_x^* \varphi \, , \, V_\xi^* U_x^* \psi \right\rangle .$$

Then, $\Theta[f; \varphi, \psi] \in BC(G \times \hat{G})$, and

$$\Theta[f_{(x,\xi)}; \varphi, \psi] = \Theta[f; U_x V_\xi \varphi, U_x V_\xi \psi]_{(x,\xi)}$$

$$\Theta[\widetilde{f_1}; \varphi, \psi](x, \xi) = \langle A_\Lambda(f) \varphi \, , \, \psi \rangle \qquad \forall f \in BC(G)$$

$$\Theta[\widetilde{g_2}; \varphi, \psi](x, \xi) = \langle B_\Gamma(g) \varphi \, , \, \psi \rangle \qquad \forall g \in BC(\hat{G}) \, .$$

Fix an invariant mean m on $G \times \hat{G}$, and define the map

$$\mathsf{M}^{\mathrm{av}} : BC(G \times \hat{G}) \to \mathcal{L}(\mathcal{H})$$
$$\langle \mathsf{M}^{\mathrm{av}}(f)\varphi \, , \, \psi \rangle = m(\Theta[f; \varphi, \psi]) \, .$$

Then, M^{av} is a operator valued mean on $G \times \hat{G}$ satisfying

$$\mathsf{M}^{\mathrm{av}}(f_{(x,\xi)}) = V_\xi^* U_x^* \mathsf{M}^{\mathrm{av}}(f) U_x V_\xi \, .$$

Moreover,

$$\begin{matrix} \mathsf{M}_1^{\mathrm{av}} = \mathsf{A}_\Lambda \\ \mathsf{M}_2^{\mathrm{av}} = \mathsf{B}_\Gamma \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \mathsf{M}_1^{\mathrm{av}}(\infty) = 0 \\ \mathsf{M}_2^{\mathrm{av}}(\infty) = 0 \end{matrix} \quad \Longrightarrow \quad \mathsf{M}^{\mathrm{av}}(\infty) = 0 \, .$$

Therefore, $\mathsf{M}^{\mathrm{av}} = \mathsf{M}_0^{\mathrm{av}}$, that is, M^{av} is an operator valued measure on $G \times \hat{G}$.

Fix an invariant mean m on $G \times \hat{G}$, and define the map

$$\mathrm{M}^{\mathrm{av}} : BC(G \times \hat{G}) \to \mathcal{L}(\mathcal{H})$$
$$\langle \mathrm{M}^{\mathrm{av}}(f)\varphi \,,\, \psi \rangle = m(\Theta[f; \varphi, \psi]) \,.$$

Then, M^{av} is a operator valued mean on $G \times \hat{G}$ satisfying

$$\mathrm{M}^{\mathrm{av}}(f_{(x,\xi)}) = V_\xi^* U_x^* \mathrm{M}^{\mathrm{av}}(f) U_x V_\xi \,.$$

Moreover,

$$\begin{array}{l} \mathrm{M}_1^{\mathrm{av}} = \mathsf{A}_\Lambda \\ \mathrm{M}_2^{\mathrm{av}} = \mathsf{B}_\Gamma \end{array} \implies \begin{array}{l} \mathrm{M}_1^{\mathrm{av}}(\infty) = 0 \\ \mathrm{M}_2^{\mathrm{av}}(\infty) = 0 \end{array} \implies \mathrm{M}^{\mathrm{av}}(\infty) = 0 \,.$$

Therefore, $\mathrm{M}^{\mathrm{av}} = \mathrm{M}_0^{\mathrm{av}}$, that is, M^{av} is an operator valued measure on $G \times \hat{G}$.

Index

1. C. Carmeli, T. Heinonen, A. Toigo, *On the coexistence of position and momentum observables*, J. Phys. A **38** (2005) 5253-5266
2. C. Carmeli, T. Heinosaari, A. Toigo, *Informationally complete joint measurements on finite quantum systems*, Phys. Rev. A **85** (2012) 012109