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Unitary representations of super groups and Mackey theory

Claudio Carmeli[1], **Gianni Cassinelli**[2], **Alessandro Toigo**[3]

[1]Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: carmeli@ge.infn.it
[2]Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: cassinelli@ge.infn.it
[3]Dipartimento di Fisica, Università di Genova, and I.N.F.N., Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy. e-mail: toigo@ge.infn.it

Abstract

We outline some basic facts in the theory of unitary representations of super Lie groups along the lines of [1]. We define the concept of representation induced from a special sub super group. We give a version of Mackey theory for groups containing a special normal abelian connected sub super group.

1 Super groups and super Harish Chandra pairs

Let $\mathcal{M}$ denote the category of super manifolds. A super Lie group (SLG) is a group object G in the category $\mathcal{M}$. This means that G is a supermanifold and that there exist morphisms:

$$\begin{aligned} m:\, G\times G &\longrightarrow G \\ i:\, G &\longrightarrow G \\ e:\, \mathbb{R}^{0|0} &\longrightarrow G \end{aligned}$$

called multiplication, inverse and unit respectively obeying the usual commutative diagrams. Due to the presence of nilpotent elements in the superalgebras of the sheaf $\mathcal{O}_G$, morphisms between super manifolds are not determined by their action on points and must be thought as morphisms of super algebras:

$$\begin{aligned} m^*:\, \mathcal{O}_G &\longrightarrow \mathcal{O}_{G\times G} \\ i^*:\, \mathcal{O}_G &\longrightarrow \mathcal{O}_G \\ e^*:\, \mathcal{O}_G &\longrightarrow \mathbb{R} \end{aligned}$$

satisfying the dualized commutative diagrams. The presence of commuting nilpotent elements in the sheaf of a topological space is at the heart of "modern" algebraic geometry. Super geometry can then be seen as a "generalization" of algebraic geometry to the case of anticommuting nilpotents. For a detailed exposition of the theory of super manifolds from this perspective we refer to [2], [8], [5], [9]. Let G_0 be the classical Lie group underlying G[1]. As in the classical case it is possible to associate to each super Lie group G the set $\mathfrak{g}$ of left invariant vector fields over it. It turns out that $\mathfrak{g}$ has the structure of a finite dimensional super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is isomorphic to the (classical) Lie algebra of G_0. For the definition and the basic properties of super Lie algebras we refer to [4] and [9]. A central role in our approach ([1]) to the representation theory of super Lie groups is played by the following result (see [2])[2]. G_0 acts naturally on $\mathfrak{g}$ through automorphisms

$$\alpha : G_0 \longrightarrow \mathrm{Aut}\,(\mathfrak{g})$$

in such a way that

$$\mathrm{d}\alpha\,(X) = [X, \cdot] \quad \forall X \in \mathfrak{g}_0$$

Then we can associate to each super Lie group G the triple $(G_0, \mathfrak{g}, \alpha)$. In what follows for simplicity we will omit the reference to α and we will use the abbreviated notation

$$(G_0, \mathfrak{g}) \tag{1}$$

the action of $g \in G_0$ on $\mathfrak{g}$ being simply denoted by $X \mapsto X^g$.

Definition 1 *We say that $(G_0, \mathfrak{g})$ is a* super Harish-Chandra pair *(SHCP) if*

- *G_0 is a classical Lie group*
- *$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super Lie algebra such that $\mathfrak{g}_0$ is isomprphic to the Lie algebra of G_0;*
- *the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}$ exponentiates to a representation of G_0.*

Remark 1 *We will refer to (1) as the SHCP defined by the super Lie group G.*

Remark 2 *Let $(G_0, \mathfrak{g})$ and $(H_0, \mathfrak{h})$ be SHCPs. We say that $\pi = (\pi_0, \rho^\pi)$ is a morphism if*

(a) *$\pi_0 : G_0 \to H_0$ is a classical Lie group homomorphism;*

[1]Let $J\,(U)$ denote the ideal generated by the nilpotent elements in $\mathcal{O}_G\,(U)$. There is a canonical isomorphism $\mathcal{O}_G\,(U)\,/J\,(U) \simeq C^\infty\,(U)$ and the maps m, i, e pass to the quotient. In this way we can define a classical Lie group $G_0 = (G, \mathcal{O}_G/J)$.

[2]A different approach to the representation theory of super Lie groups can be found in [3]

(b) $\rho^\pi : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of super Lie algebras such that*

$$\begin{aligned} \rho^\pi|_{\mathfrak{g}_0} &= \mathrm{d}\pi_0 \\ \rho^\pi(X^g) &= \rho^\pi(X)^g \end{aligned}$$

We thus see that super Harish-Chandra pairs form a category.

Example 1 *Let $V = V_0 \oplus V_1$ be a super vector space. The SHCP defined by the SLG $GL(V)$ is given by*

$$\begin{aligned} G_0 &= GL(V_0) \times GL(V_1) \\ \mathfrak{g} &= \mathbf{End}(V) \end{aligned}$$

where $GL(V_i)$ is the classical Lie group of linear isomorphisms of V_i, $\mathbf{End}(V)$ is the super Lie algebra of linear transformations of V, and the actions are the natural ones.

Theorem 1 *The assignement that takes a super Lie group G into the corresponding SHCP $(G, \mathfrak{g})$ is an equivalence of categories.*

By virtue of the above theorem in the following we will make no distinction between a super Lie group and the corresponding SHCP.

A finite dimensional representation of a SLG G in a super vector space V, i.e. a super Lie group homomorphism $\pi : G \to GL(V)$, is thus equivalent to a morphism (π_0, ρ^π) between the respective SHCPs, i.e.

1. an *even* (= grading preserving) representation π_0 of the classical Lie group G_0 in V;
2. a super Lie algebra homomorphism $\rho^\pi : \mathfrak{g} \to \mathbf{End}(V)$ satisfying

$$\begin{aligned} \rho^\pi|_{\mathfrak{g}_0} &= \mathrm{d}\pi_0 \\ \rho^\pi(X^g) &= \pi_0(g)\rho^\pi(X)\pi_0(g)^{-1}. \end{aligned}$$

Actually, condition 2 is equivalent to give a linear map $\rho^\pi : \mathfrak{g}_1 \to \mathbf{End}(V)_1$ such that

$$\begin{aligned} \mathrm{d}\pi_0([X,Y]) &= \rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X) \quad \forall X, Y \in \mathfrak{g} \\ \rho^\pi(X^g) &= \pi_0(g)\rho^\pi(X)\pi_0(g)^{-1}. \end{aligned}$$

This definition can be generalized to the infinite dimensional case. Nevertheless the relation

$$\rho^\pi(X)^2 = \mathrm{d}\pi_0([X,X])$$

makes apparent that great care is needed due to the appearance of unbounded operators.

Let us restrict from now on to the case of unitary representations. If G is a classical Lie group and π is a strongly continuous unitary representation (in brief, UR) of G, it is a classical result that the generators $\mathrm{d}\pi\,(X)$, $X \in \mathfrak{g}$, are skew-adjoint, and the vector space $C^\infty\,(\pi)$ of infinitely differentiable vectors for π is a common dense invariant core for the whole set of operators $\{\mathrm{d}\pi\,(X)\}_{X\in\mathfrak{g}}$.

The following definition is completely natural.

Definition 2 *A* super Hilbert space *(SHS) is a super vector space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ over $\mathbb{C}$ with a scalar product $(\cdot,\cdot)$ such that $\mathcal{H}$ is a Hilbert space under $(\cdot,\cdot)$, and $\mathcal{H}_i$ $(i = 0, 1)$ are mutually orthogonal closed linear subspaces. If we define*

$$\langle x, y\rangle = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are of opposite parity} \\ (x, y) & \text{if } x \text{ and } y \text{ are even} \\ i(x, y) & \text{if } x \text{ and } y \text{ are odd} \end{cases}$$

then $\langle x, y\rangle$ is an even super Hermitian form, *with*

$$\langle y, x\rangle = (-1)^{p(x)p(y)}\overline{\langle x, y\rangle}.$$

We give the following definition of unitary representation.

Definition 3 *A* unitary representation *(UR) of a SLG $(G_0, \mathfrak{g})$ is a triple $(\pi_0, \rho^\pi, \mathcal{H})$ with the following properties.*

(a) $\mathcal{H}$ is a SHS, and π_0 is an even UR of G_0 in $\mathcal{H}$;

(b) $\rho^\pi : \mathfrak{g}_1 \to \mathbf{End}(C^\infty(\pi_0))_1$ is a linear map such that

(i) *$\rho^\pi(X)$ with domain $C^\infty(\pi_0)$ is symmetric for all $X \in \mathfrak{g}_1$;*

(ii) *$\rho^\pi(X^g) = \pi_0(g)\rho^\pi(X)\pi_0(g)^{-1}$ for all $X \in \mathfrak{g}_1, g \in G_0$;*

(iii) *$-i\mathrm{d}\pi_0([X, Y]) = \rho^\pi(X)\rho^\pi(Y) + \rho^\pi(Y)\rho^\pi(X)$ on $C^\infty(\pi_0)$ for all $X, Y \in \mathfrak{g}_1$.*

Note that we have included a factor $e^{-i\pi/4}$ in ρ^π. It may appear that the choice of $C^\infty\,(\pi_0)$ in the above definition is somewhat arbitrary. It turns out that all such choices are essentially equivalent in the following sense.

Proposition 1 *Let be given*

(a) *an even unitary representation π_0 of G_0 in a SHS $\mathcal{H}$;*

(b) *a π_0-invariant dense graded subspace $\mathcal{B} \subset \mathcal{H}$;*

(c) *a linear map $\rho : \mathfrak{g}_1 \to \mathbf{End}(\mathcal{B})_1$ such that*

(i) *$\rho(X)$ is a symmetric operator for all $X \in \mathfrak{g}_1$;*

(ii) *$\rho(X^g) = \pi_0(g)\rho(X)\pi_0(g)^{-1}$ for all $X \in \mathfrak{g}_1, g \in G_0$;*

(iii) *$-i\mathrm{d}\pi_0([X, Y]) = \rho(X)\rho(Y) + \rho(Y)\rho(X)$ on $\mathcal{B}$ for all $X, Y \in \mathfrak{g}_1$.*

Then

1. *For any* $X \in \mathfrak{g}_1$, $\rho(X)$ *is essentially self adjoint and* $C^\infty(\pi_0) \subset \mathrm{dom}\overline{\rho(X)}$.

2. *Let* $\rho^\pi(X) = \overline{\rho(X)}\big|_{C^\infty(\pi_0)}$ *for* $X \in \mathfrak{g}_1$. *Then* $(\pi_0, \rho^\pi, \mathcal{H})$ *is a* UR *of the* SLG $(G_0, \mathfrak{g})$.

Moreover, $(\pi_0, \rho^\pi, \mathcal{H})$ *is unique, in the sense that if* $(\pi_0, \rho', \mathcal{H})$ *is a* UR *of the* SLG $(G_0, \mathfrak{g})$, *such that* $\mathcal{B} \subset \mathrm{dom}\overline{\rho'(X)}$ *and* $\overline{\rho'(X)}$ *restricts to* $\rho(X)$ *on* $\mathcal{B}$ *for all* $X \in \mathfrak{g}_1$, *then* $\rho' = \rho^\pi$.

With the following natural definition of morphism between URs, Schur lemma follows easily.

Definition 4 *A* morphism $A : \pi \to \pi'$, π *and* π' *being URs of* $(G_0, \mathfrak{g})$, *is an* even *bounded operator* $A : \mathcal{H} \to \mathcal{H}'$ *such that*

(a) $A\pi_0(g) = \pi_0'(g)A$ *for all* $g \in G_0$*;*

(b) $A\rho^\pi(X) = \rho^{\pi'}(X)A$ *for all* $X \in \mathfrak{g}_1$, *where* $AC^\infty(\pi_0) \subset C^\infty(\pi_0')A$ *follows by* (a).

We let $\mathrm{Hom}\,(\pi, \pi')$ *be the set of morphisms* $\pi \to \pi'$.

Proposition 2 (Schur lemma) *If* π, π' *are irreducible (i.e. they do not contain subrepresentations), then* $\mathrm{Hom}\,(\pi, \pi') = \mathbb{C}$ *or* 0.

2 Mackey theory for super groups

One of the main procedure for determining the unitary dual of a classical group is given by Mackey theory [7]. At the heart of such theory there are the concepts of induced representation and imprimitivity system. In this section we will sketch very briefly such notions in the super context and, in order to simplify notations, we will assume all groups at sight *unimodular* and, due to lack of space, we will usually omit measure theoretical details. For a detailed exposition we refer to [1].

2.1 The inducing functor and the super imprimitivity theorem

Let $G = (G_0, \mathfrak{g})$ be a SHCP and $H = (H_0, \mathfrak{h})$ a sub-SHCP, i.e. H_0 is a *closed* subgroup of G_0 and $\mathfrak{h} \subset \mathfrak{g}$. We want to define the concept of representation of G induced by a representation $\sigma = (\sigma_0, \rho^\sigma, \mathcal{K})$ of H. We will restrict our treatment to the case in which H have a particular form.

Definition 5 H *is called a* special *sub SHCP if* $\mathfrak{h}_1 = \mathfrak{g}_1$.

Remark 3 *In particular, this condition implies*

1. $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{h}_0$.

2. *the quotient of G with respect to H is a classical manifold.*

The first step in defining the representation π of $(G_0, \mathfrak{g})$ induced by σ is to define the space $\mathcal{H}$ on which π acts. This is completely classical. We briefly summarize the main points of this construction. We denote by $g \mapsto \dot{g}$ the canonical projection of G_0 onto G_0/H_0, let also μ be a fixed invariant measure on G_0/H_0. $\mathcal{H}$ is defined as the set of those functions $f : G_0 \to \mathcal{K}$ such that

1. f is weakly measurable;
2. $f(gh) = \sigma_0(h)^{-1} f(g)$;
3. $\|f\|_{\mathcal{K}} \in L^2(G_0/H_0, \mu)$.

The unitary action π_0 of G_0 in $\mathcal{H}$ is by left translations. π_0 so defined is the classical representation of G_0 induced by σ_0.

Remark 4 *$\mathcal{H}$ is naturally graded, $f \in \mathcal{H}$ being even or odd according as $f(g) \in \mathcal{K}_0$ or $f(g) \in \mathcal{K}_1$ for μ-almost all $\dot{g}$.*

In order to define the action of $\mathfrak{g}_1$ we need to determine, as explicitely as possible, $C^\infty(\pi_0)$. It turns out [10] that

$$C^\infty(\pi_0) = \{f \in \mathcal{H} \cap C^\infty(G_0; \mathcal{K}) \mid Df \in \mathcal{H} \text{ for all left invariant differential operators } D\}$$

It is important to note that that the vectors in $C^\infty(\pi_0)$ take values in $C^\infty(\sigma_0)$. The following definition is then well posed.

$$(\rho^\pi(X) f)(g) = \rho^\sigma\left(X^{g^{-1}}\right) f(g) \quad \forall X \in \mathfrak{g}_1,\ f \in C^\infty(\pi_0)$$

It turns out (see [1]) that $\rho^\pi(X)$ maps $C^\infty(\pi_0)$ into itself.

We call the representation $(\pi_0, \rho^\pi, \mathcal{H})$ defined above the representation of G *induced* by σ, and we denote it by $\mathrm{ind}(\sigma)$.

It is well known that attached to the classical representation π_0 induced by σ_0 there is a projection valued measure $P : \mathcal{B}(G_0/H_0) \to \mathcal{L}(\mathcal{H})$ ($\mathcal{B}(G_0/H_0) =$ the Borel subsets of G_0/H_0, $\mathcal{L}(\mathcal{H}) =$ the bounded operators on $\mathcal{H}$) such that

$$\pi_0(g) P(E) \pi_0(g)^{-1} = P(gE) \quad \forall E \in \mathcal{B}(G_0/H_0).$$

P is given by

$$(P(E)f)(g) = \chi_E(g) f(g) \tag{2}$$

($\chi_E =$ the characteristic function of E). The triple $(\pi_0, P, \mathcal{H})$ is the classical system of imprimitivity induced by σ_0. It turns out that $P(E)$ commutes with $\overline{\rho^\pi(X)}$ on $\mathrm{dom}\overline{\rho^\pi(X)}$. This justifies the following definition.

Definition 6 *A* super system of imprimitivity *(SSI) for G based on G/H is a collection $(\pi_0, \rho^\pi, P, \mathcal{H})$ where*

(i) $\pi = (\pi_0, \rho^\pi, \mathcal{H})$ *is a UR of the SLG* $(G_0, \mathfrak{g})$*;*

(ii) $(\pi_0, P, \mathcal{H})$ *is a classical system of imprimitivity;*

(iii) $P(E)$ *is an even operator for all* $E \subset \mathcal{B}(G_0/H_0)$*;*

(iv) $\overline{\rho^\pi(X)}$ *commutes with* $P(E)$ *for all* $X \in \mathfrak{g}_1$ *and* $E \subset \mathcal{B}(G_0/H_0)$.

The SSI $(\pi_0, \rho^\pi, P, \mathcal{H})$ *with* $\pi = \operatorname{ind}(\sigma)$ *and* P *given by eq. 2 is the SSI* induced *by* σ.

Morphisms between SSIs based on G_0/H_0 are defined in the natural way. We thus have the following extension of Mackey's imprimitivity theorem.

Theorem 2 (Super imprimitivity theorem) *The assignment that takes* σ *to the SSI induced by* σ *is an equivalence of categories from the category of URs of the special sub SLG* $H = (H_0, \mathfrak{h})$ *to the category of SSIs for* G *based on* G/H.

2.2 Mackey machine

Let now G be a SLG and N a special sub SLG. Suppose N_0 is a connected abelian Lie group, and that N_0 is normal in G_0. Denote by $\widehat{N}_0$ the unitary dual of N_0. Let $\pi = (\pi_0, \rho^\pi, \mathcal{H})$ be an *irreducible* UR of G. Since N_0 is abelian, there exists a unique projection valued measure $P : \mathcal{B}\left(\widehat{N}_0\right) \to \mathcal{L}(\mathcal{H})$ such that

$$\pi(n) = \int_{\widehat{N}_0} \xi(n) \mathrm{d}P(\xi) \quad \forall n \in N_0.$$

It is not difficult to check that

$$\pi_0(g) P(E) \pi_0(g)^{-1} \quad = \quad P\left(E^g\right).$$

From now on, we will suppose that N_0 *acts trivially* on $\mathfrak{g}_1$. Due to the relation

$$\rho^\pi\left(X\right) = \rho^\pi\left(X^n\right) = \pi_0(g)\rho^\pi(X)\pi_0(g)^{-1} \quad \forall n \in N,\, X \in \mathfrak{g}_1,$$

we see that $\overline{\rho^\pi(X)}$ commutes with $P(E)$ for all $E \in \mathcal{B}\left(\widehat{N}_0\right)$. Hence, if E is G_0-invariant, $P(E) \in \operatorname{Hom}(\pi, \pi)$. By Schur lemma, we conclude that $P(E) = 0$ or I.

If the orbit space $\widehat{N}_0/G_0$ is *countably separated* we have that P is concetrated in an orbit $\mathcal{O} \subset \widehat{N}_0$. In this case, fixed $\xi \in \mathcal{O}$ and denoting by $G_{0\xi}$ the stability subgroup of ξ in G_0, the orbit $\mathcal{O}$ is homeomorphic to the quotient space $G_0/G_{0\xi}$, and P transports to a projection valued measure on $G_0/G_{0\xi}$. Let $G_\xi = (G_{0\xi}, \mathfrak{g}_{0\xi} \oplus \mathfrak{g}_1)$, which we call the *stabilizer* of ξ in G. G_ξ is a special sub-SHCP of G containing N. We have thus shown that $(\pi_0, \rho^\pi, P, \mathcal{H})$ is a SSI for G based on G/G_ξ. By super imprimitivity theorem, each SSI arising in this manner from an irreducible representation π of G is induced by a representation

$\sigma = (\sigma_0, \rho^\sigma, \mathcal{K})$ of G_ξ. It is a classical result (see [6]) that such representation must satisfy

$$\sigma_0(n) = \xi(n)I \quad \forall n \in N_0. \tag{3}$$

We call ξ-*admissible* a representation σ of G_ξ satisfying eq. 3, and denote by $\check{G}_\xi$ the set of irreducible ξ-admissible representations of G_ξ. We have thus proved the following theorem.

Theorem 3 *Let N be a sub-SHCP of Gsuch that*

(i) *N is special;*

(ii) *N_0 is a normal abelian and connected;*

(iii) *the action of N_0 on $\mathfrak{g}_1$ is trivial;*

(iv) *the orbit space N_0^*/G_0 is countably separated.*

then

(i) *fixed $\xi \in \widehat{N}_0$, for each $\sigma \in \check{G}_\xi$ the representation* $\mathrm{ind}\,(\sigma)$ *is irreducible, and* $\mathrm{ind}\,(\sigma) \neq \mathrm{ind}\,(\sigma')$ *if $\sigma \neq \sigma'$;*

(ii) *each irreducible representation of G is of the form* $\mathrm{ind}\,(\sigma)$ *for some $\xi \in \widehat{N}_0$ and $\sigma \in \check{G}_\xi$;*

(iii) *inducing from a different $\xi' \in \widehat{N}_0$ gives the same set of representations if ξ and ξ' are in the same orbit, and disjoint sets if ξ and ξ' are in different orbits.*

We notice that here an important difference arise with respect to the classical theory since for some $\xi \in \widehat{N}_0$ there can be no ξ-admissible representation. We call $\mathcal{O}^+$ the G_0-invariant set of those $\xi \in \widehat{N}_0$ for which $\check{G}_{0\xi}$ is non empty. The above proposition establishes a bijective correspondence between the unitary dual $\hat{G}$ of the SLG G and the fibered set $\cup_{\xi \in \mathcal{O}^+/G_0} \check{G}_\xi$, in complete analogy with the classical theory. However, we stress again that, unlike to the classical case, in the super case *not all orbits are allowed*.

Determination of $\check{G}_{0\xi}$ In this last section, given $\xi \in \widehat{N}_0$ we will determine the set $\check{G}_\xi$ of ξ-admissible irreducible representations of G_ξ. We define the simmetric bilinear form $Q_\xi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$ given by

$$Q_\xi\,(X, Y) = -i\mathrm{d}\xi\,([X, Y])\,.$$

Suppose $\sigma = (\sigma_0, \rho^\sigma, \mathcal{K}) \in \check{G}_\xi$. Since $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{n}_0$,

$$Q_\xi\,(X, X)\,I = -i\mathrm{d}\sigma_0\,([X, X]) = 2\rho^\sigma\,(X)^2\,, \tag{4}$$

thus showing that Q_ξ is semidefinite positive, and that ρ^σ extends to a bounded self-adjoint operator on $\mathcal{K}$. We denote by $\mathfrak{g}_{1\xi}$ the quotient of $\mathfrak{g}_1$ by the radical of

Q_ξ. By eq. 4, the map $X \mapsto \rho^\sigma(X)$ passes to the quotient $\mathfrak{g}_{1\xi}$. We let $\mathcal{C}(\mathfrak{g}_{1\xi})$ be the Clifford algebra of $\mathfrak{g}_{1\xi}$ (with respect to the scalar product induced by Q_ξ). The relation

$$Q_\xi(X,Y) = \rho^\sigma(X)\rho^\sigma(Y) + \rho^\sigma(Y)\rho^\sigma(X)$$

implies that ρ^σ extends to a *self-adjoint representation* (SAR) of the superalgebra $\mathcal{C}(\mathfrak{g}_{1\xi})$, i.e. to a *graded* representation of $\mathcal{C}(\mathfrak{g}_{1\xi})$ by bounded operators on $\mathcal{K}$ whose restriction to $\mathfrak{g}_{1\xi}$ acts by self-adjoint operators.

We now give a brief analysis of these kind of representations. In [1] the following facts are proved.

- (a) There exists irreducible SARs of $\mathcal{C}(\mathfrak{g}_{1\xi})$; these are finite-dimensional, unique if $\dim \mathfrak{g}_{1\xi}$ is odd, and unique up to parity reversal if $\dim \mathfrak{g}_{1\xi}$ is even.
- (b) Let τ be an irreducible SAR of $\mathcal{C}(\mathfrak{g}_{1\xi})$ in a SHS $\mathcal{L}$ and let θ be any SAR of $\mathcal{C}(\mathfrak{g}_{1\xi})$ in a SHS $\mathcal{R}$. Then $\mathcal{R} = \mathcal{M} \otimes \mathcal{L}$, where $\mathcal{M}$ is a SHS and
$$\theta(x) = 1 \otimes \tau(x) \quad \forall x \in \mathcal{C}(\mathfrak{g}_{1\xi}).$$
Moreover, if $\dim \mathfrak{g}_{1\xi}$ is odd, $\mathcal{M}$ can be chosen purely even.
- (c) If τ is an irreducible SAR of $\mathcal{C}(\mathfrak{g}_{1\xi})$, the restriction of τ to the spin group $\mathrm{Spin}\,(\mathfrak{g}_{1\xi}) \subset \mathcal{C}(\mathfrak{g}_{1\xi})$ is unitary.

For $g \in G_{0\xi}$, we have

$$Q_\xi([X^g, Y^g]) = -i\mathrm{d}\xi([X,Y]^g) = -i\mathrm{d}\xi([X,Y]),$$

which shows that the action of $G_{0\xi}$ (actually, of the quotient group $G_{0\xi}/N_0$) on $\mathfrak{g}_1$ descends to an action on $\mathfrak{g}_{1\xi}$ by orthogonal transformations. We now assume that the stability subgroup $G_{0\xi}$ is *connected* (for a complete treatement, we refer to [1]). In this case, $G_{0\xi}/N_0$ maps into $SO(\mathfrak{g}_{1\xi})$, so, for each $\dot{g} \in G_{0\xi}/N_0$, we can find a (not unique!) $\tilde{g} \in \mathrm{Spin}\,(\mathfrak{g}_{1\xi})$ such that

$$\dot{X}^g = \tilde{g}\dot{X}\tilde{g}^{-1} \quad \forall \dot{X} \in \mathfrak{g}_{1\xi}. \tag{5}$$

If τ is an irreducible SAR of $\mathcal{C}(\mathfrak{g}_{1\xi})$, we define

$$\kappa(\dot{g}) := \tau(\tilde{g}). \tag{6}$$

We thus get a projective unitary representation κ of $G_{0\xi}/N_0$, with ±1-valued multiplier μ, which by eq. 5 satisfies

$$\kappa(\dot{g})\tau(\dot{X})\kappa(\dot{g})^{-1} = \tau(\dot{X}^g).$$

We now return to the problem of determining the structure of $\sigma \in \check{G}_{0\xi}$. By the above facts, we can write

$$\rho^\sigma(X) = 1 \otimes \tau_\xi(\dot{X})$$

for some irreducible SAR τ_ξ of $\mathcal{C}\left(\mathfrak{g}_{1\xi}\right)$. Following the classical pattern, we will show that σ_0 can be written as a tensor product of "simpler" projective representations.

It is well known that the character ξ of N_0 cannot be in general extended to a UR of $G_{0\xi}$ (Mackey obstruction; see [6]). Nevertheless it is always possible to extend ξ to a *projective* scalar representation $\tilde{\xi}$ of $G_{0\xi}$. The class of the associated multiplier ω_ξ depends only on ξ, and ω_ξ descends to a multiplier of the group $G_{0\xi}/N_0$. If we define the representation

$$\tilde{\sigma}(g) = \tilde{\xi}(g)^{-1}\sigma(g)$$

it is a straightforward computation to check that $\tilde{\sigma}$ is a $\overline{\omega}_\xi$-projective representation. Moreover

$$\tilde{\sigma}\left(gn\right) = \xi(n)^{-1}\tilde{\xi}\left(g\right)^{-1}\sigma(g)\xi(n)$$

and hence $\tilde{\sigma}$ descends to a $\overline{\omega}_\xi$-projective representation of $G_{0\xi}/N_0$, which we denote again by $\tilde{\sigma}$.

Now $\tilde{\sigma}$ must satisfy

$$\tilde{\sigma}(\dot{g})\left(1\otimes\tau_\xi(\dot{X})\right)\tilde{\sigma}(\dot{g})^{-1} = \sigma_0(g)\rho^\sigma\left(X\right)\sigma_0(g)^{-1} = \rho^\sigma\left(X^g\right).$$

Let κ_ξ be as in eq. 6. Since $\left[1\otimes\kappa_\xi(\dot{g})\right]^{-1}\tilde{\sigma}(\dot{g})$ commutes with $1\otimes\tau_\xi$, by irreducibility of τ_ξ we conclude that

$$\tilde{\sigma}(\dot{g}) = \overline{\sigma}(\dot{g})\otimes\kappa_\xi(\dot{g})$$

where $\overline{\sigma}$ is an *even* $\overline{\omega}_\xi\mu$ representation of $G_{0\xi}/N_0$ in $\mathcal{M}$.

We have finally established the structure of a generic admissible irreducible UR $\sigma\in\check{G}_{0\xi}$, namely

$$\begin{aligned}\sigma_0(g) &= \tilde{\xi}(g)\overline{\sigma}(\dot{g})\otimes\kappa_\xi(\dot{g})\\ \rho^\sigma(X) &= 1\otimes\tau_\xi(\dot{X})\end{aligned}$$

where $\tilde{\xi}$, κ_ξ and τ_ξ are fixed by the point $\xi\in\widehat{N}_0$, and $\overline{\sigma}\in\widehat{G_{0\xi}/N_0}^{\overline{\omega}\mu}$, the set of $\overline{\omega}\mu$-projective representations of $G_{0\xi}/N_0$. In particular, this shows that $Q_\xi\geq 0$ is not only necessary, but is also a sufficient condition in order that $\check{G}_{0\xi}\neq\emptyset$. In fact, $Q_\xi\geq 0$ is the only condition needed to carry over the above construction.

We have thus established the following result.

Theorem 4 *We have the identification*

$$\mathcal{O}^+ = \left\{\xi\in\widehat{N}_0 \mid Q_\xi\geq 0\right\}.$$

If $\xi\in\mathcal{O}^+$, fix an irreducible SAR τ_ξ of $\mathcal{C}\left(\mathfrak{g}_{1\xi}\right)$. Then, the map

$$\widehat{G_{0\xi}/N_0}^{\overline{\omega}\mu}\ni\overline{\sigma}\longmapsto\left(\tilde{\xi}\overline{\sigma}\otimes\kappa_\xi, 1\otimes\tau_\xi\right)\in\check{G}_{0\xi}$$

is a bijection.

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