UNIQUENESS AND RECONSTRUCTION PROBLEMS IN GEOMETRIC TOMOGRAPHY

Background. Image reconstruction from projections is one of the main inverse problems which appears in several applications. The image is usually represented by an unknown real valued function $f(x, y)$, with bounded support. The values of f are related to physical properties of a two-dimensional section of the object under investigation. Projections are taken with the help of some kind of rays. For instance, in Computerized Tomography (CT), a portion of a human body is reconstructed by measuring the coefficient of linear attenuation of each beam of the X-ray traveling along a line crossing the body. The radiation is produced by photons, issued from a source and collected by a detector, both translating and rotating around the body. The differences between issued and collected photons measure the absorption of radiation by different tissues.
Mathematical approach. The underlying mathematical approach is known since 1917, and goes back to J. Radon, who described a direct method for inverting the so-called Radon Transform (RT) of f to get the density function $f(x, y)$ of a planar section K of the body. We shall briefly mention the basic ingredients.

To specify a line of photons in the plane we use two coordinates: r, its distance from the origin, and θ, the angle that the line of detectors (orthogonal to the lines of photons) makes with the positive x-axis. Then, a single photon on the line has coordinates

$$\begin{cases} x = r\cos\theta - s\sin\theta \\ y = r\sin\theta + s\cos\theta, \end{cases}$$

where the parameter $s \in \mathbb{R}$ identifies a photon on its line (see Figure 1).
For each $r \in \mathbb{R}$ and $0 \le \theta < \pi$, the collected information is given by the integral of f along a line of photons crossing the body:

$$p_\theta(r)(K) = \int_{-\infty}^{+\infty} f(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds.$$

Thus, for each $r \in \mathbb{R}$ the Radon Transform of f is given by

$$(\mathcal{R}f) = \{p_\theta(r)(K) \,:\, 0 \le \theta < \pi\}.$$

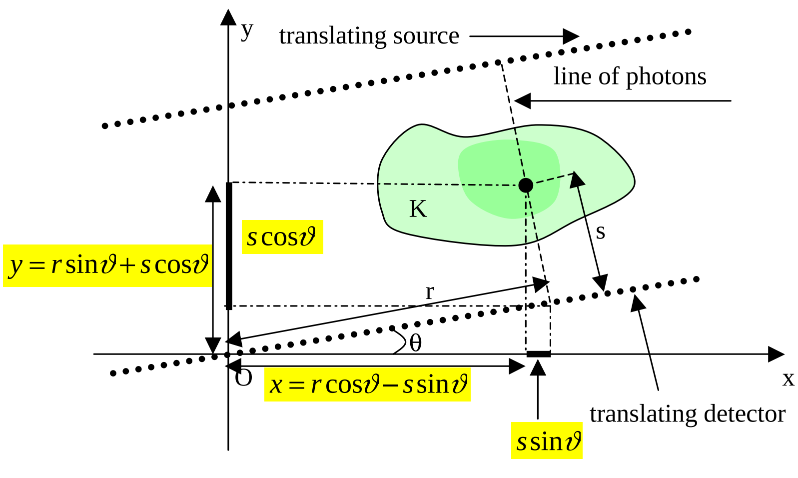

FIGURE 1. Collection of data under a rotation $\theta \in [0, \pi)$.

The main tool for the inversion of the RT is the so-called *central slice theorem* (or *projection slice theorem*), which says that the restriction to $u = r\cos\theta, v = r\sin\theta$ of the 2-dimensional Fourier transform $\hat{f}(u, v)$ of $f(x, y)$ equals the 1-dimensional Fourier transform of $p_\theta(r)$. From the knowledge of $(\mathcal{R}f)$, $p_\theta(r)$ is known for all θ, and thus $\hat{f}$, so that f can be determined by means of the inverse Fourier Transform [24].

Problems in applications. Despite this clear theoretical picture, many problems remain in applications, where the main effort is the discretization of the whole process, due to different reasons. To begin with, only a finite number of projections can be taken, i.e. only a finite number of directions can be considered. Moreover, for each direction the X-ray is not continuous, but actually consists of a finite number of beams. This implies that one can collect only a finite number of data $p_{\theta_i}(r_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$. Being the numerical data discrete, the density function f must be redefined on a finite gride, so that the inverse problem itself becomes discrete. Consequently, the implementation of any reconstruction algorithm in CT produces gray-scale digital images, where every gray level has a finite binary representation. This leads to a natural connection between CT and Discrete Tomography.

Geometric Tomography. Geometric Tomography (GT) is a geometric relative of CT. In GT the usual density functions appearing in CT are replaced by geometric objects, and one of the main goal is to find conditions which guarantee a faithful reconstruction, possibly unique, within a given geometric class of subsets of $\mathbb{R}^n$. The parallel X-ray of a homogeneous set K, in a given direction θ, is represented by the length of each chord of K parallel to θ. Higher dimensional X-rays (k-dimensional planes replacing chords) or source X-rays (issuing from a point in a fixed position) can also be considered. Presence of ambiguities is often related to the so-called switching-components, or analogous geometric configurations, and uniqueness results are mainly obtained by their analysis. Usually considered classes of geometric objects are represented by convex sets and by star-shaped sets.

Hammer's Problem. In 1963, P.C. Hammer asked [19]: *How many parallel X-ray pictures of a convex body must be taken in order to permit its exact reconstruction?* The following basic example (see [18]) shows that there are finite sets of directions, with arbitrary large cardinality, such that the corresponding X-rays do not distinguish a convex body among the others. Consider a regular q-gon Q centred at a fixed point p, and its rotation Q' by π/q about p. The convex hull of Q and Q' is a $2q$-gon and let θ be a direction parallel to one of its edges. It is easy to see that Q and Q' have the same parallel X-rays in the direction θ. It is important to note that Hammer's problem is one of an affine nature, since a nonsingular affine transformation preserves the ratios of lengths of parallel line segments. Thus, for uniqueness of the reconstruction one should avoid subsets of directions of the edges of an affinely regular polygon, i.e. the image of a regular polygon under some affine transformation.

R.J. Gardner and P. McMullen proved in [17] that *convex bodies are determined by X-rays taken in any set of directions that is not a subset of the directions of the edges of an affinely regular polygon.* Since the cross ratio of any four directions of the edges of a regular polygon is an algebraic number, any set of four directions with a transcendental cross ratio uniquely determines a convex body by means of the corresponding X-rays.

It might be objected that two congruent polygons, as well as their affine images, are not really that different at all. By answering this objection, for any set of directions of the edges of a regular $2n$-gon A. Volčič [27] constructed a family, with cardinality equal to that of the real numbers, of mutually non congruent convex bodies all with the same X-rays in the considered directions, see also [15, Theorem 1.2.13]. Motivated by genuine applications in the material sciences, Hammer's problem has been studied recently within discrete tomography. In this field, important and deep results have been obtained by R. Gardner and P. Gritzmann, which imply the following surprising result for continuous X-rays (see [16, Theorem 6.2]): convex bodies in the Euclidean plane are determined by their parallel X-rays in *any* set of seven mutually nonparallel lattice directions. See also [8] for a description of some geometric properties related to ambiguities, and [7] for the investigation of the same problem by means of source X-rays in the integer lattice.

Uniqueness by means of additivity. The class of additive sets has been widely considered too. Let H be a subspace of $\mathbb{R}^n$. A *ridge function orthogonal to* H is a function which is constant on each translate of H.

Let $\mathcal{H} = \{H_i : 1 \leq i \leq m\}$ be a set of subspaces of $\mathbb{R}^n$ of dimension between 1 and $n-1$ inclusive. A bounded set $E \subset \mathbb{R}^n$ is called $\mathcal{H}$-*additive* if

$$E = \{x \in \mathbb{R}^n : \sum_i f_i(x) > 0\}, \tag{0.1}$$

where f_i is a ridge function orthogonal to H_i (see [15, Chapter 2], where a slightly more general definition is given). The following theorem is proved in [13].

Theorem 1. *Any $\mathcal{H}$-additive set is uniquely reconstructible by means of X-rays parallel to the subspaces in $\mathcal{H}$.*

In the planar case, the additivity property has an interesting interplay with convexity, and, in particular, with the notion of *inscribability.* A convex body $K \subset \mathbb{R}^2$ is said to be inscribable with respect to a finite set $\mathcal{D}$ of directions, or simply $\mathcal{D}$-*inscribable*, if its interior is the union of interiors of convex polygons inscribed in K, each of whose edges is parallel to some direction in $\mathcal{D}$. If $\mathcal{D}$ consists of the set of coordinate directions in the plane, then $\mathcal{D}$-inscribability and uniqueness by means of X-rays in the directions in $\mathcal{D}$ are equivalent (see [20]). Further, in this case, a planar convex body K is $\mathcal{D}$-inscribable if and only if it is $\mathcal{D}$-additive (see [14]). This result partially extends to any finite set $\mathcal{D}$ of directions: every $\mathcal{D}$-inscribable set is $\mathcal{D}$-additive, but the converse is not true (see [14] and also [15, Chapter 1] for an overview on this topic).

Stability estimates. Usually, in the applications, data contain errors, and uniqueness and stability estimates are crucial in order to show when Hammer's problem is well posed. A. Volčič proved in [26] that the reconstruction of a convex set K is well posed when the set of directions guarantees uniqueness. Roughly speaking, if we know the parallel X-rays of K in a finite set of directions, with the unique determination, and the data X-rays contain an error ε, then the corresponding reconstruction K_ε converges to K when ε tends to zero.

Denoting by $|K \triangle K_\varepsilon|$ the area of the symmetric difference of K and K_ε, the L_2-distance of the difference of the characteristic functions of K and K_ε measures how K_ε is close to the desired solution K. Such a distance is usually compared with the L_2-distance of the corresponding X-rays data and with the number n of data. The order of stability for this distance is known explicitly in the class of smooth (not necessarily convex) domains and it is of order $1/2$ with respect to ε and $1/n$ (see [25]).

When the parallel X-rays of K are known up to an error ε for any planar direction and under the a-priori assumption that the planar body is convex (not necessarily smooth), as considered in Hammer's problem, such order of stability estimate is improved to 1 with respect to ε (see [23]).

It is worth remarking that the uniqueness results obtained in [17] and in [16] are, unfortunately, unstable in the sense that a small perturbation of a finite set of directions providing uniqueness may cause the uniqueness property to be lost, so that the above results of well-posedness cannot be used. In view of this, it is of interest to investigate further stability estimates for non uniqueness situations. Observe that in the basic example considered above the convex q-gons Q and Q' are close enough for q sufficiently large. In particular the symmetric difference $Q \triangle Q'$ has area of order 2 with respect to q^{-1}, so that the distance $|Q \triangle Q'|^{1/2}$ is invariant under equi-affine transformation and it is of order 1. It can be shown (see [12]) that any non uniqueness situation in Hammer's problem has the same upper bound for the area distance.

Related aspects.

Graph Theory. From a graph-theory point of view, the ambiguous configurations can be described as follows. We have a bipartite graph $G(V, E)$ together with a set of points S, such that $V = A \cup B$, $|A| = |B|$, and for each pair $v \in V$, and $s \in S$, the line through v and s meets a single further vertex $w \in V$ in the complement of the component of v. Connections between Tomography and Graph Theory can also be found by means of suitable notions of RT for a graph. An example can be given in terms of paths passing through the vertices of the graph [6]. This leads to some generalization of graph isomorphisms called path-congruences (see [9, 10, 11]), and to typical reconstruction problems.

Two-dimensional languages. The inverse problems considered in Tomography are closed also to questions coming from pattern recognition and image processing. The increasing interest on these topics has motivated the research on two-dimensional (2D for short) languages. Here, words can be considered as pictures or 2D-arrays of symbols, chosen in a finite alphabet. The main attempt is to generalize the richness of the theory of 1D-languages to two dimensions. References [1, 2, 3, 4, 5] represent recent works of our group in this direction.

References

[1] M. Bersani, A.Cherubini, A. Frigeri, *On Some Classes of 2D Languages and Their Relations*, Lecture Notes in Computer Science 6636 (IWCIA 2011), 222-234.

[2] A. Cherubini, S. Crespi-Reghizzi, M.Pradella, *Regional Languages and Tiling: A Unifying Approach to Picture Grammars*, Lecture Notes in Computer Science 5162 (MFCS 2008), 253-264.

[3] A. Cherubini, S. Crespi-Reghizzi, M.Pradella, *A unifying approach to picture grammars*, Inf. Comput. 209(9)(2011), 1246-1267.

[4] A. Cherubini, S. Crespi-Reghizzi, M.Pradella, Pierluigi San Pietro, *Picture languages: Tiling systems versus tile rewriting grammars*, Theor. Comput. Sci. 356(1-2): 90-103 (2006).

[5] A. Cherubini, M.Pradella, *Picture Languages: From Wang Tiles to 2D Grammars*, Lecture Notes in Computer Science 5725 (CAI 2009), 13-46.

[6] P. Dulio, *Geometric Tomography in a Graph*, Rend. Circ. Mat. Palermo, (2) Suppl., 77 (2006), 229-266.

[7] P. Dulio, R.J. Gardner and C.Peri, *Discrete point X-rays*, SIAM J. Discrete Math. 20, no. 1 (2006), 171-188.

[8] P. Dulio and C. Peri, On the geometric structure of lattice U-polygons, Discrete Math., 307/19-20 (2007), 2330-2340.

[9] P. Dulio and V. Pannone, *Trees with the same path-table*, Le Matematiche, (Catania) **60** (2005), no. 1, 59-65 (2006).

[10] P. Dulio and V. Pannone, *Trees with path-stable center*, Ars Combinatoria, LXXX (2006), 153-175.

[11] P. Dulio and V. Pannone, *Joining caterpillars and stability of the tree center*, Discrete Math., 308/7, (2008), 1185-1190.

[12] P. Dulio P, M. Longinetti, C. Peri C and A. Venturi, *Sharp affine stability estimates for Hammer's problem*, Advances in Applied Mathematics, 41/1, (2008), 27-51

[13] P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp, *Sets uniquely determined by projections on axes I. Continuous case*, SIAM J. Appl. Math. 50 (1990), pp. 288-306.

[14] R. J. Gardner, *Sets determined by finitely many X-rays*, Geom. Dedicata 43 (1992), pp. 1–6.

[15] R. J. Gardner, *Geometric Tomography*, 2nd ed. Cambridge University Press, New York, 2006.

[16] R. J. Gardner and P Gritzmann, Discrete Tomography: Determination of finite sets by X-rays, Transactions Amer. Math. Soc. 349 (1997) 2271-2295.

[17] R. J. Gardner and P. McMullen, On Hammer's X-ray Problem, J. London Math. Soc. 21 (2) (1980) 171-175.

[18] O. Giering, Bestimmung von Eibereichen und Eikörpen durch Steiner-Symmetrisierungen, Sber. Bayer. Akad. Wiss. München, Math.-Nat.Kl. (1962) 225-253.

[19] P. C. Hammer, Problem 2, in: V.L. Klee (Ed.), Proc. Symp. in Pure Mathematics, vol. VII: Convexity, American Mathematical Society, Providence, RI, 1963, pp. 498-499.

[20] A. Kuba and A. Volčič, *Characterization of measurable plane sets which are reconstructible from their two projections*, Inverse Problems, 4, (1988), pp. 513-27.

[21] M. Longinetti, An Isoperimetric inequality for convex polygons and convex sets with the same symmetrals, Geom. Dedicata 20 (1986) 27-41.

[22] M. Longinetti, Una proprietà di massimo dei poligoni affinemente regolari, Rend. Circ. Mat. Palermo (2) 34 (3) (1985 448-459.

[23] M. Longinetti, Some questions of stability in the reconstruction of plane convex bodies from projections, Inverse Problems 1 (1) (1985) 87-97.

[24] F. Natterer, *The Mathematics of Computerized Tomography*, SIAM, Philadelphia, 2001.

[25] A.K. Louis and F. Natterer, Mathematical Problems of Computerised Tomography, Proc of the IEEE, 71 (3) (1983) 379-389.

[26] A. Volčič , Well-posedness of the Gardner-McMullen reconstruction problem, Proc. Conf. Measure Theory, Oberwolfach 1983, Lecture Notes in Mathematics 1089, Springer, Berlin, 1984, pp. 199-210.

[27] A. Volčič , Ghost convex bodies, Boll. Un. Mat. Ital. A (6) 4 (2) (1985) 287-292.